Quantum transport

TIGP course
Advanced Nanotechnology (A)

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Bibliography

- Supriyo Datta, *Electric Transport in Mesoscopic Systems* (1995)
- Douglas Natelson, *Nanostructures and Nanotechnology* (2015)
- Thierry Giamarchi, *Quantum Physics in One Dimension* (2003)
- Additional references listed in the slides

Recruitment

- Quantum Matter Theory Group at IoP, AS
- We welcome postdocs, assistants, and students to join us
- Welcome to share the information!

In this class …

- goal: concept of quantum transport
- materials:
  - single-particle regime:
    - Douglas Natelson, *Nanostructures and nanotechnology* (Sec. 6.4)
      (unfortunately some typos …)
    - Supriyo Datta, *Electronic Transport in Mesoscopic Systems*
      (older but still useful)
  - beyond the single-particle regime:
    - Thierry Giamarchi, *Quantum Physics in One Dimension*
    - additional references on interacting 1D systems
- warning: inconsistent notations from different sources

Outline

- Review of useful concepts from quantum mechanics
- Quantum transport in mesoscopic systems
  - Landauer-Büttiker formalism (single-particle description)
    - conductance quantization in ballistic systems
    - Landauer formula for an imperfect conductor
    - Büttiker formula for multiterminal devices
    - application
  - interacting systems (beyond single-particle regime)
    - interacting electrons in 1D: Tomonaga-Luttinger liquid
    - impurities (weak and strong)
    - effects of spin-orbit-coupling

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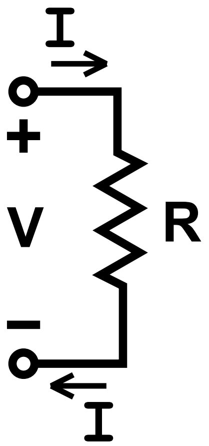
Charge transport in macroscopic devices

- Ohm's law:

$$V = IR$$



Wikipedia page of Ohm's law

R : resistance, V : voltage, I : current

- resistance (geometry) vs resistivity (material)

$$R = \rho \frac{L}{A}$$

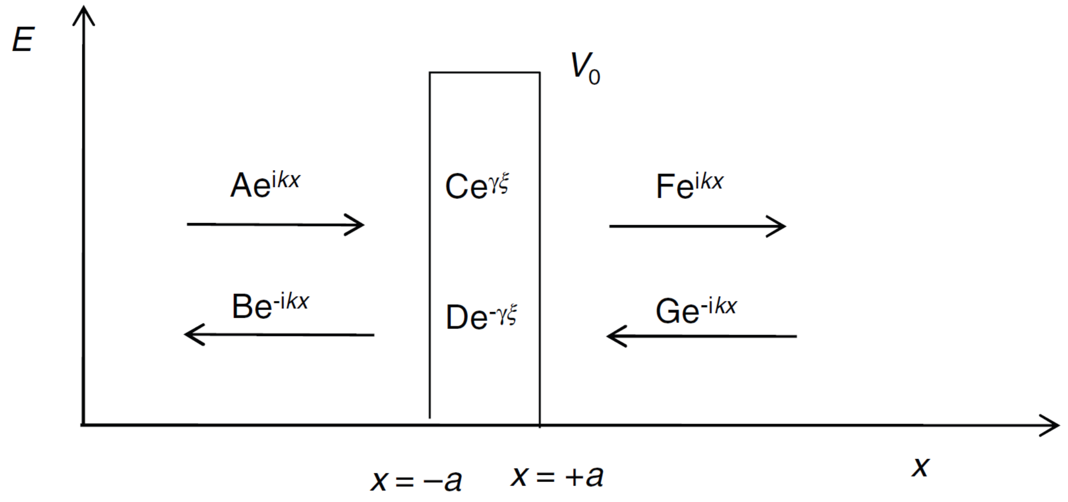
ρ : resistivity, L : length of a conductor, A : cross-section area

- question: what if we shrink the conductor ($A \searrow$) so that there can be only few electrons passing through it at a time?
    - typical length scales $\sim O(100 \text{ nm}) \Rightarrow$ comparable to the size of coronavirus!
    - recall: how electrons move in simple 1D potential

Electron tunneling through a barrier

- Schrödinger equation with effective m : $i\hbar\partial_t\psi(x,t) = \left[-\frac{\hbar^2}{2m}\,\partial_x^2 + V(x)\right]\psi(x,t)$

- for time-independent potential: $\left[-\frac{\hbar^2}{2m}\,\partial_x^2 + V(x)\right]\phi(x) = E\,\phi(x)$
- tunneling through a barrier with square potential (height V_0 and width $2a$)



[note] typo in Fig. 6.14

general solution for $\mathrm{E} < V_0$:

$$\phi(x) = \begin{cases} \mathrm{A}\mathrm{e}^{\mathrm{i}kx} + B\mathrm{e}^{-\mathrm{i}kx}, & x < -a, \\ C\mathrm{e}^{\gamma x} + D\mathrm{e}^{-\gamma x}, & -a < x < a, \\ F\mathrm{e}^{\mathrm{i}kx} + G\mathrm{e}^{-\mathrm{i}kx}, & x > a, \end{cases}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad \gamma = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

Tunneling through a barrier

- boundary conditions: continuous $\phi(x)$ and $\partial_x\phi(x)$ at $x = \pm a$:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{ik+\gamma}{2ik} e^{(ik-\gamma)a} & \frac{ik-\gamma}{2ik} e^{(ik+\gamma)a} \\ \frac{ik-\gamma}{2ik} e^{-(ik+\gamma)a} & \frac{ik+\gamma}{2ik} e^{-(ik-\gamma)a} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}, \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \frac{ik+\gamma}{2\gamma} e^{(ik-\gamma)a} & -\frac{ik-\gamma}{2\gamma} e^{(ik+\gamma)a} \\ -\frac{ik-\gamma}{2\gamma} e^{-(ik+\gamma)a} & \frac{ik+\gamma}{2\gamma} e^{-(ik-\gamma)a} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} F \\ G \end{pmatrix}, \ \boldsymbol{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ik+\gamma}{2ik} e^{(ik-\gamma)a} & \frac{ik-\gamma}{2ik} e^{(ik+\gamma)a} \\ \frac{ik-\gamma}{2ik} e^{-(ik+\gamma)a} & \frac{ik+\gamma}{2ik} e^{-(ik-\gamma)a} \end{pmatrix} \begin{pmatrix} \frac{ik+\gamma}{2\gamma} e^{(ik-\gamma)a} & -\frac{ik-\gamma}{2\gamma} e^{(ik+\gamma)a} \\ -\frac{ik-\gamma}{2\gamma} e^{-(ik+\gamma)a} & \frac{ik+\gamma}{2\gamma} e^{-(ik-\gamma)a} \end{pmatrix}$$

- transfer matrix ***M*** : describing how a particle tunnels through the barrier
  - diagonal terms: transmission through the barrier

    $\Rightarrow$ related to charge current

Current density and transmission coefficient

- particle current density from quantum mechanics:

$$J = \frac{\hbar}{2mi}\,(\phi^*\partial_x\phi \; - \; \phi\partial_x\phi^*)$$

- left side of the barrier ($x < -$a):

$$J_< = \frac{\hbar k}{m}\,(|A|^2 - |B|^2)$$

- right side of the barrier ($x >$ a):

$$J_> = \frac{\hbar k}{m}\,(|F|^2 - |G|^2)$$

- for a particle coming from $x = -\infty$ ($G \to 0$), the probability that it passes through the barrier and that it gets reflected:

$$T(E) \equiv |F|^2/|A|^2 = 1/|M_{11}|^2, \quad R(E) \equiv |B|^2/|A|^2$$

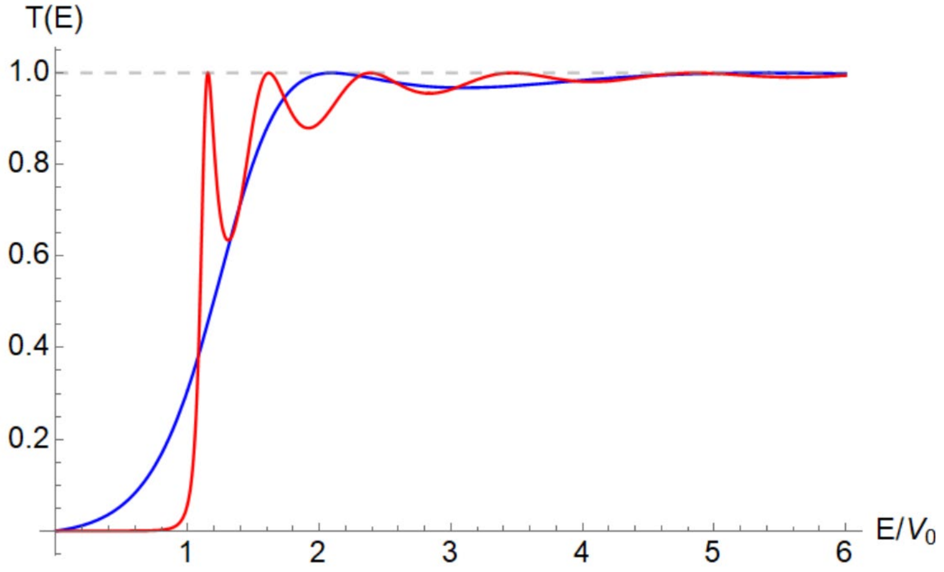
- current related to transmission probability and element of *M*

Transmission coefficient

$$E < V_0 : \; T(E) = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)}$$

$$E > V_0 : \; T(E) = \frac{1}{1 + \frac{V_0^2}{4E(E - V_0)} \sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E - V_0)}\right)}$$

[note] typo in Eq. (6.43)



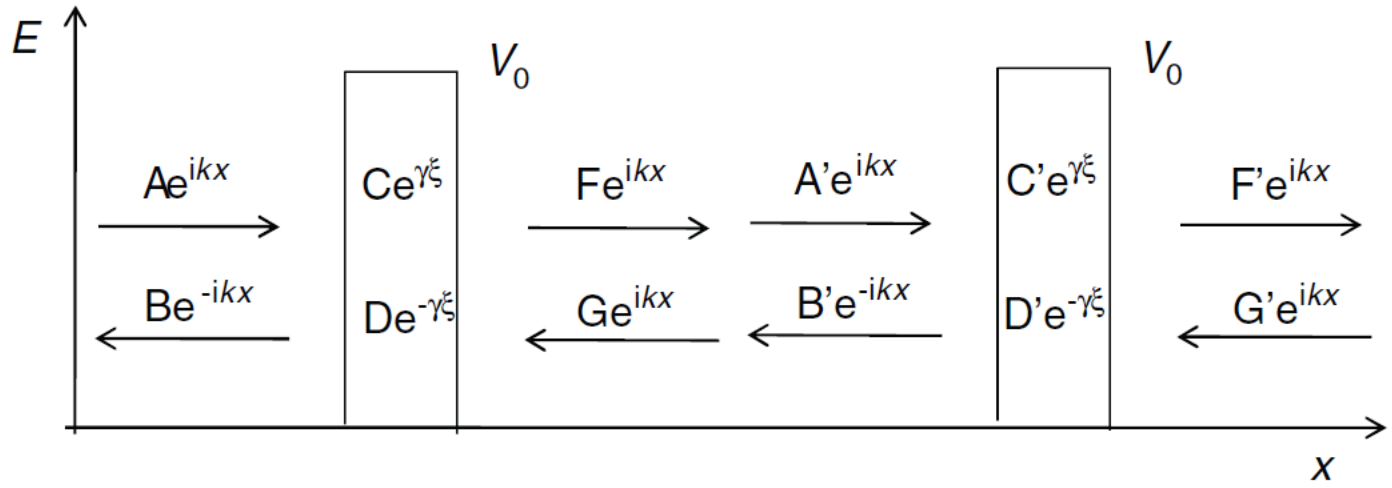
- nonzero probability for $E < V_0$ (classically forbidden regime)
- weak-tunneling regime (a wide and/or tall barrier, $\gamma a \gg 1$)

$$T(E) \approx \frac{4E(V_0 - E)}{V_0^2} e^{-\frac{4a}{\hbar}\sqrt{2m(V_0 - E)}} \propto e^{-\frac{4a}{\hbar}\sqrt{2m(V_0 - E)}}$$

⇒ exponential dependence on the barrier thickness

⇒ sensitivity useful for STM/STS

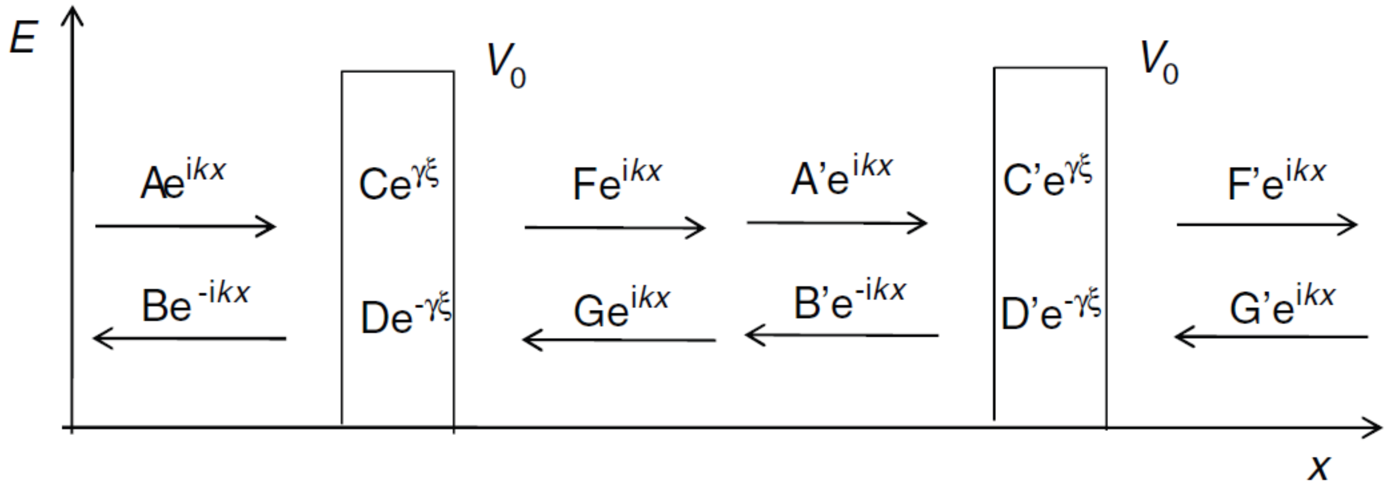
Double-barrier tunneling



$$\begin{pmatrix} A \\ B \end{pmatrix} = \boldsymbol{M_L} \begin{pmatrix} F \\ G \end{pmatrix}, \qquad \begin{pmatrix} F \\ G \end{pmatrix} = \boldsymbol{M_W} \begin{pmatrix} A' \\ B' \end{pmatrix}, \qquad \begin{pmatrix} A' \\ B' \end{pmatrix} = \boldsymbol{M_R} \begin{pmatrix} F' \\ G' \end{pmatrix}$$

- $\boldsymbol{M_{L(R)}}$: tunneling through the left (right) barrier
- $\boldsymbol{M_W}$: propagation in the well with inter-barrier distance *b* $\quad \boldsymbol{M_W} = \begin{pmatrix} e^{-ikb} & 0 \\ 0 & e^{ikb} \end{pmatrix}$

Double-barrier tunneling (conti.)



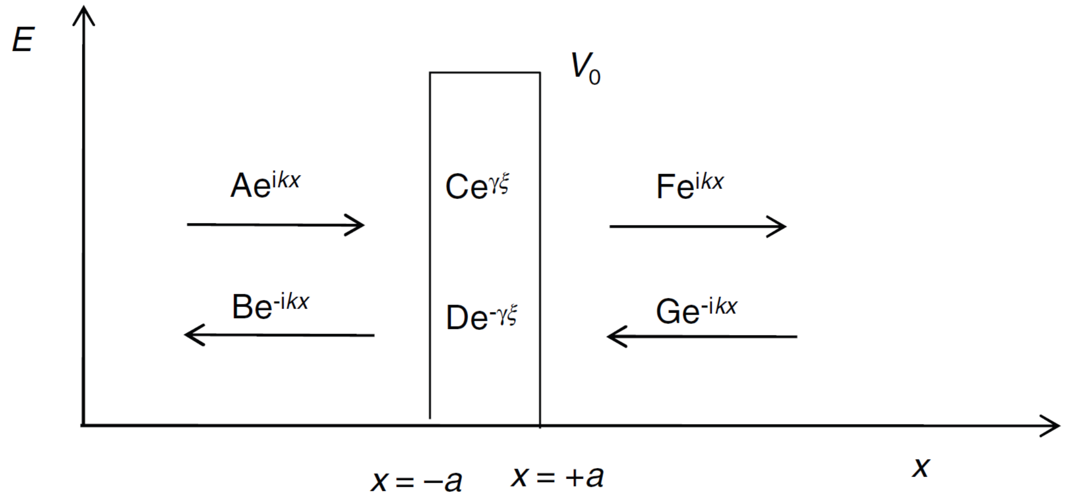
- total transmission coefficient

$$\boldsymbol{M_{tot}} = \boldsymbol{M_L M_W M_R}, \quad T_{tot}(E) = 1/\left|M_{tot,11}\right|^2$$ **[note] typo in Eq. (6.50)**

- resonance condition with transmission probability = 1
- generalized for multiple barriers via multiplication of $\boldsymbol{M}$ matrix

Scattering matrix formalism

- instead of expressing $\phi(x)$ for $x < -a$ in terms of that for $x > a$, we can express the outgoing wave in terms of the incoming wave



$$\begin{pmatrix} A \\ B \end{pmatrix} = \boldsymbol{M} \begin{pmatrix} F \\ G \end{pmatrix}, \qquad \boldsymbol{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} B \\ F \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} A \\ G \end{pmatrix}, \qquad \boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

- the role of $\boldsymbol{M}$ replaced by the scattering matrix $\boldsymbol{S}$
- transmission probability in terms of the element of $\boldsymbol{S}$: $T(E) = |S_{12}|^2$

Two scattering regions

- combining scattering matrices in regions 1 & 2:

$$\boldsymbol{S_{tot}} = \boldsymbol{S}^{(1)} \otimes \boldsymbol{S}^{(1-2)} \otimes \ \boldsymbol{S}^{(2)}$$

- $\boldsymbol{S}^{(1-2)}$: how regions 1 & 2 are connected
- $\otimes$: combining $\boldsymbol{S}^{(1)}$ & $\boldsymbol{S}^{(2)}$ in a way depending on their coherence
  - full coherence: combining amplitudes (elements of $\boldsymbol{S}$)

    given $\begin{pmatrix} r_1 & {t_1}' \\ t_1 & {r_1}' \end{pmatrix}$ and $\begin{pmatrix} r_2 & {t_2}' \\ t_2 & {r_2}' \end{pmatrix} \Rightarrow T_{tot} = \left|\frac{t_1 t_2}{1-r_1' r_2}\right|^2 = \frac{T_1 T_2}{1-2\sqrt{R_1 R_2}\cos\theta + R_1 R_2}$ (resonance!)

$$T_{1,2} = \left|t_{1,2}\right|^2 = \left|t'_{1,2}\right|^2 \quad R_{1,2} = \left|r_{1,2}\right|^2 = \left|r'_{1,2}\right|^2$$

$$\theta = \text{phase}\,(r_1') + \text{phase}\,(r_2)$$

  - complete decoherence: probability instead of amplitude

    given $\begin{pmatrix} R_1 & T_1 \\ T_1 & R_1 \end{pmatrix}$ and $\begin{pmatrix} R_2 & T_2 \\ T_2 & R_2 \end{pmatrix} \Rightarrow T_{tot} = \frac{T_1 T_2}{1-R_1 R_2}$ (no resonance)
  - partial coherence: modeling with fictitious leads

Scattering matrix for multiple modes

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & \cdots & S_{1N} \\ \vdots & \ddots & \vdots \\ S_{N1} & \cdots & S_{NN} \end{pmatrix}$$

- matrix $\boldsymbol{S}$: unitary (ensured by current conservation)
- transmission from mode n to mode m : $T_{m\leftarrow n} = |S_{m\leftarrow n}|^2$
- important concept for the development of Büttiker formula

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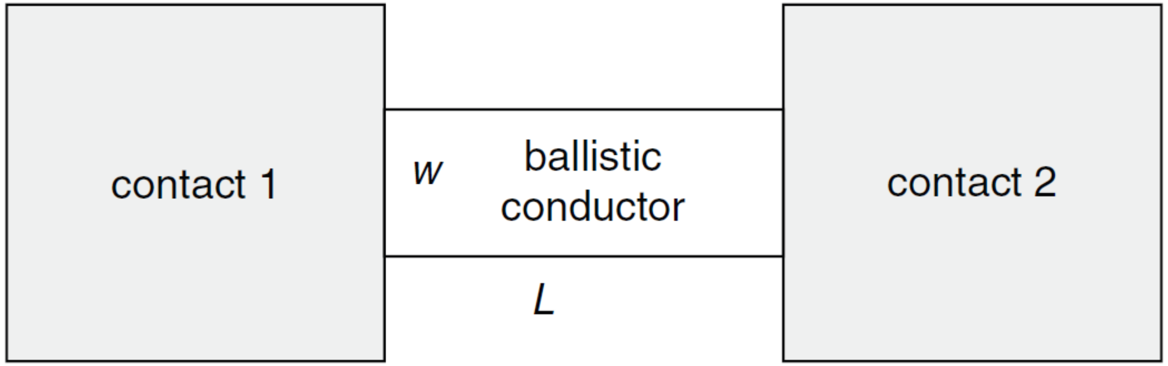
Transport through a ballistic conductor

- *ballistic conductor*:

  a 1D or (quasi-1D) system with length L much shorter than the mean free path of carriers (no scattering inside)

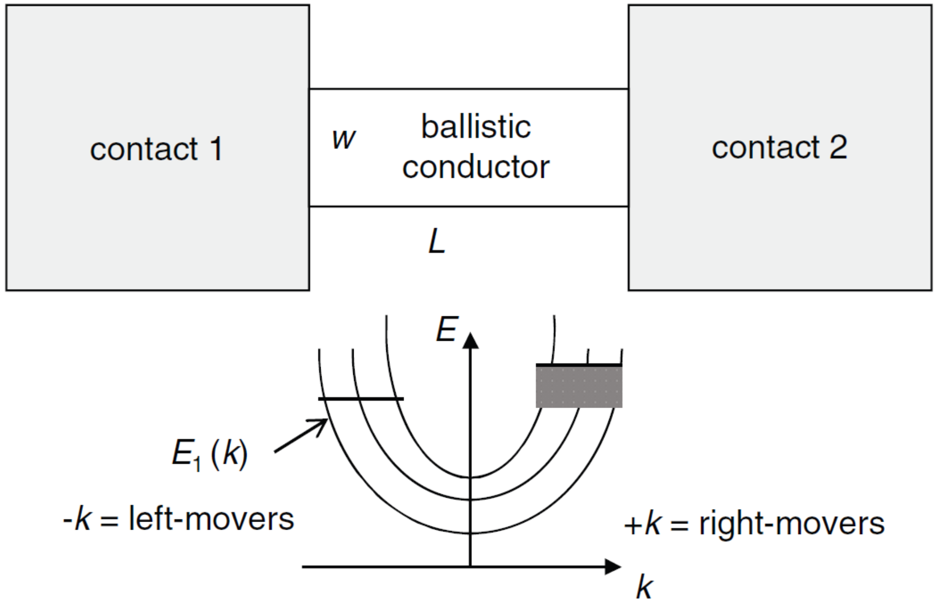
- quantum coherent scattering region connected via contacts to classical reservoirs



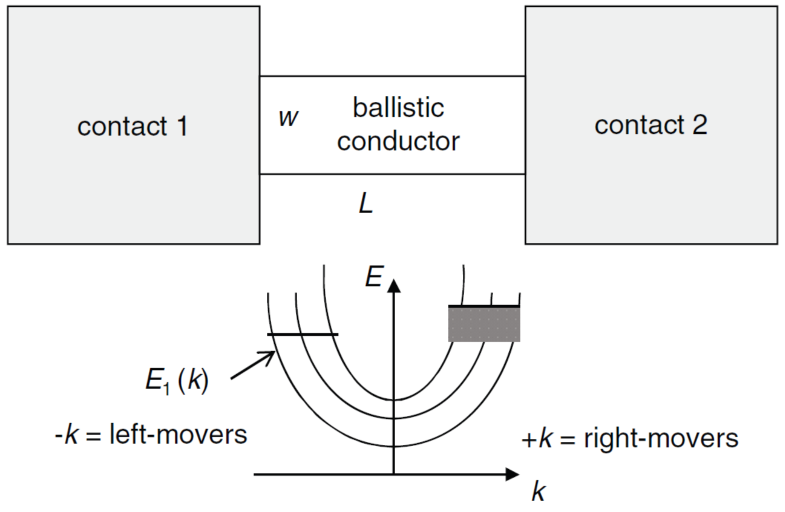
- no reflection within the conductor

  $\Rightarrow$ transmission probability = 1

Electric conductance of a ballistic conductor



- Q1D conductor with M transverse modes
- at $T = 0$: electrons filled up to μ_L (μ_R) in the left (right) contact
- number of modes with energies $E_j < E$: $M(E) = \sum_j \Theta\left(E - E_j\right)$
- $M = 3$ here



- difference in μ_L and μ_R set by a bias $V = \frac{\mu_L - \mu_R}{-e}$
- occupation prob. for right-moving carriers

$$f_+(E, T, \mu_L)$$

  - current from left to right:

$$I_+ = -\frac{e}{L}\sum_k v\big(E(k)\big) f_+(E, T, \mu_L)\, M\big(E(k)\big) \rightarrow -\frac{2e}{h}\int_0^\infty dE\, f_+(E, T, \mu_L)\, M(E)$$

- occupation prob. for left-moving carriers

$$f_-(E, T, \mu_R)$$

  - current in the opposite direction (from right to left):

$$I_- = -\frac{e}{L}\sum_k v\big(E(k)\big) f_-(E, T, \mu_R)\, M\big(E(k)\big) \rightarrow -\frac{2e}{h}\int_0^\infty dE\, f_-(E, T, \mu_R)\, M(E)$$

- net current at $T=0$:

$$I = I_+ - I_- = -\frac{2e}{h}\int_0^\infty dE[\, f_+(E,T,\mu_L) - f_-(E,T,\mu_R)\,]\, M(E)$$
$$= -\frac{2e}{h} M(\mu_L - \mu_R) = \frac{2e^2}{h} MV$$

- 2-terminal conductance of a M-channel ballistic conductor:

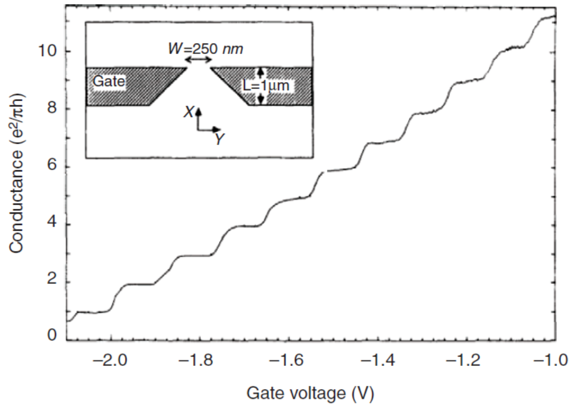
$$G = \frac{dI}{dV} = \frac{2e^2}{h} M$$

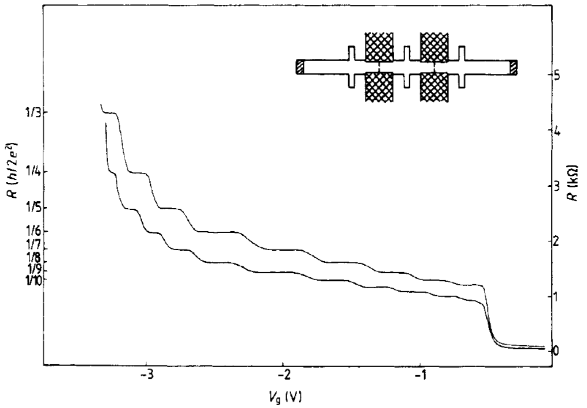
- contact resistance: $1/G = \dfrac{h}{2e^2}\dfrac{1}{M}$
  - physical meaning:

    resistance arising from the process where most of the electron wave packet from a 3D reservoir (a large number of modes) gets reflected when trying to enter a Q1D conductor (a few conduction modes) ⇒ the contact resistance arises at the interface!
  - apart from the number of channels, the contact resistance is given by universal constants (independent of material parameters)!

Conductance quantization in mesoscopic devices

- formation of conductance plateaus at $G = \frac{2e^2}{h} \times M$



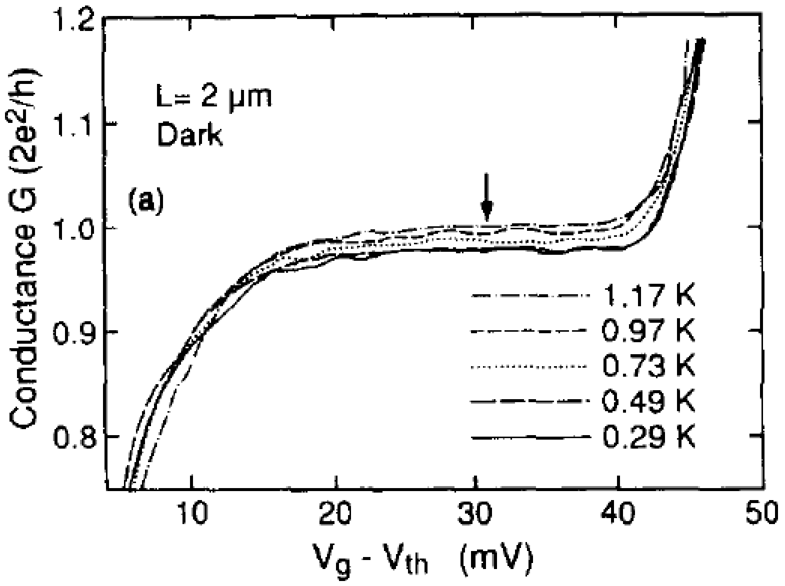


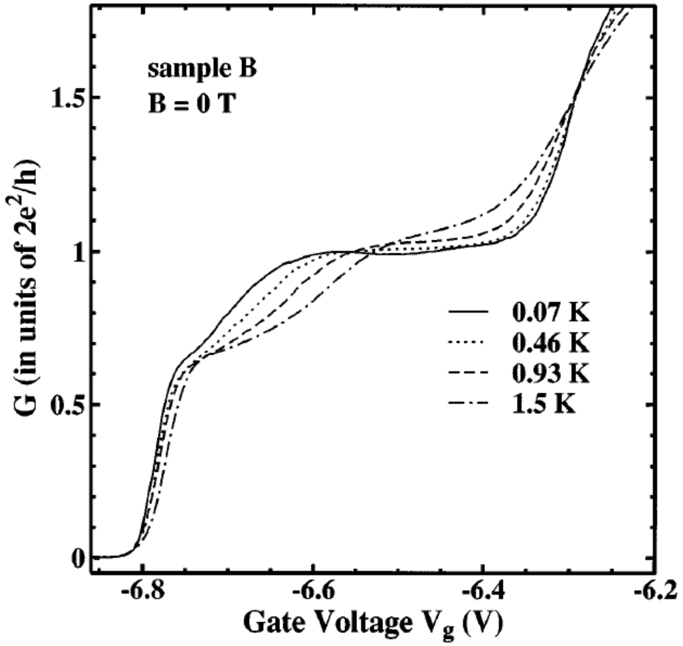
van Wees et al., PRL 60, 848 (1988)

Wharam et al., J. Phys. C: Sol. State Phys. 21, L209 (1988)

- observed in a gate-defined quantum point contact (QPC) formed in semiconductors
- voltage applied to gates to pinch off the constriction
- lowest plateau: anomaly at $0.7 \times \left(\frac{2e^2}{h}\right)$
- macroscopic scale: $G = \sigma W/L$ vs mesoscopic scale: $G = \frac{2e^2}{h} \times M$

Additional features on top of quantization





Tarucha et al., Sol. State. Commun. 94, 413 (1995)   Thomas et al., PRL 77, 135 (1996)

- uniformly reduced conductance plateau(s)
  ⇒ *e-e* interaction + disorder (discussed later)
- shoulder-like feature at $0.7 \times \left(\frac{2e^2}{h}\right)$ in the lowest plateau
  ⇒ 0.7 anomaly (spin effects? *e-e* interaction? fractionalization?)

Landauer formula

- for an imperfect conductor with multiple transverse modes
- 2-terminal conductance of a M-channel imperfect conductor:

$$G = \frac{2e^2}{h} M\,\bar{T}$$

- $\bar{T}$: transmission coefficient through a scatterer/impurity
  (assumed to be energy-independent between μ_L and μ_R)
- resistance of a conductor containing a scatterer:

$$1/G = \frac{h}{2e^2}\frac{1}{M\bar{T}} = \left(\frac{h}{2e^2}\frac{1}{M}\right) + \left(\frac{h}{2e^2}\frac{1}{M}\frac{1-\bar{T}}{\bar{T}}\right)$$

$\Rightarrow$ total resistance of a "circuit" consisting of contact resistance and scatterer-induced resistance in series

Resistance contributions from more scatterers

- total resistance of a conductor with a scatterer:

$$1/G = \frac{h}{2e^2}\frac{1}{M\bar{T}} = \frac{h}{2e^2}\frac{1}{M} + \frac{h}{2e^2}\frac{1}{M}\frac{1-\bar{T}}{\bar{T}}$$

- how about a conductor containing 2 scatterers?
  - probability of a particle passing through both scatterers

(x) $\bar{T}_1\bar{T}_2$

(o) $\bar{T}_{12} = \bar{T}_1\bar{T}_2 + \bar{T}_1 R_2 R_1 \bar{T}_2 + \bar{T}_1 R_2 R_1 R_2 R_1 \bar{T}_2 + \cdots$

$$= \bar{T}_1\bar{T}_2 + \bar{T}_1\bar{T}_2 R_1 R_2 + \bar{T}_1\bar{T}_2 R_1^2 R_2^2 + \cdots = \bar{T}_1\bar{T}_2\frac{1}{1-R_1R_2}$$ (incoherently)

$$\Rightarrow \frac{1-\bar{T}_{12}}{\bar{T}_{12}} = \frac{1-\bar{T}_1}{\bar{T}_1} + \frac{1-\bar{T}_2}{\bar{T}_2}$$

- total resistance of a conductor with 2 scatterers: $1/G = \frac{h}{2e^2}\frac{1}{M} + \frac{h}{2e^2}\frac{1}{M}\frac{1-\bar{T}_1}{\bar{T}_1} + \frac{h}{2e^2}\frac{1}{M}\frac{1-\bar{T}_2}{\bar{T}_2}$

Recovering Ohm's scaling for a long conductor

- resistance of a M-channel conductor with a scatterer:

$$R = \frac{h}{2e^2}\frac{1}{M} + \frac{h}{2e^2}\frac{1}{M}\frac{1-\bar{T}}{\bar{T}}$$

- for 2 scatterers:

$$R = \frac{h}{2e^2}\frac{1}{M} + \frac{h}{2e^2}\frac{1}{M}\frac{1-\bar{T}_1}{\bar{T}_1} + \frac{h}{2e^2}\frac{1}{M}\frac{1-\bar{T}_2}{\bar{T}_2}$$

- for a long conductor with many scatterers:

$$1/G = \frac{h}{2e^2}\frac{1}{M} + \frac{h}{2e^2}\frac{1}{M}\sum_n \frac{1-\bar{T}_n}{\bar{T}_n}$$

- assuming N scatterers with the same transmission coefficient $\bar{T}_n \to \bar{T}_1$:

$$\frac{1-\bar{T}_N}{\bar{T}_N} = \sum_n \frac{1-\bar{T}_n}{\bar{T}_n} \to N\frac{1-\bar{T}_1}{\bar{T}_1} \quad \Rightarrow \bar{T}_N = \frac{\bar{T}_1}{N(1-\bar{T}_1)+\bar{T}_1} \to \frac{L_0}{L+L_0}$$

- resistance of a long conductor with many scatterers: $R \propto \frac{h}{2e^2}\frac{1}{M}\frac{1}{\bar{T}_N} \propto \frac{L}{W}$

Effects of disorder on transport

- in realistic systems, disorder or charge impurities are (omni)present
- they induce a random potential

$$V_{\rm dis}(x) = \sum_q V_{{\rm dis},q}\, e^{iqx}\,, \qquad V_{{\rm dis},q}\text{: Fourier component of the potential}$$

- coupling to charge density $\rho \;= \sum_\sigma \psi_\sigma^\dagger\, \psi_\sigma$ with the electron field operator

$$\psi_\sigma \approx e^{ik_F x} R_\sigma + e^{-ik_F x} L_\sigma$$

- entering the Hamiltonian as a perturbation term:

$$H_{\rm dis} = \int dx\, V_{\rm dis}(x)\, \rho(x)$$

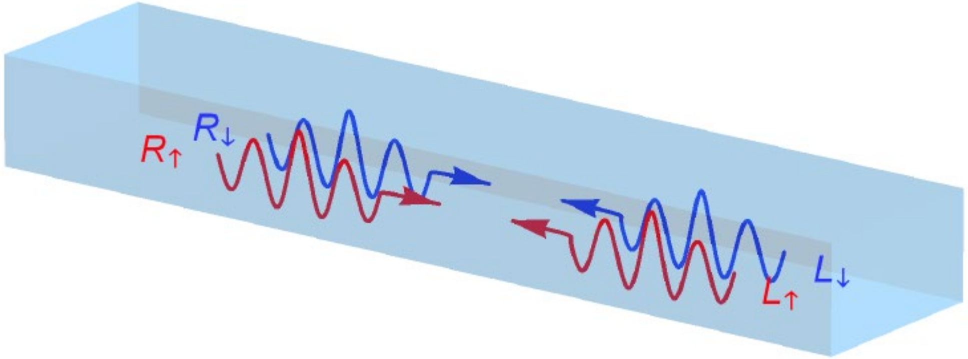
$$= \int dx\, V_{\rm dis}(x)\, (R_\sigma^\dagger R_\sigma + L_\sigma^\dagger\, L_\sigma + e^{-2ik_F x} R_\sigma^\dagger L_\sigma + e^{2ik_F x} L_\sigma^\dagger\, R_\sigma)$$

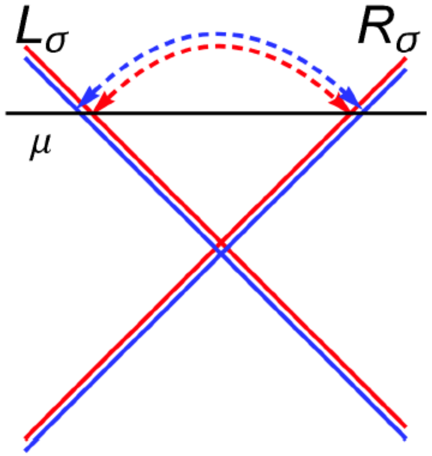
$\Rightarrow$ forward scattering of electrons: $R_\sigma^\dagger R_\sigma$, $L_\sigma^\dagger L_\sigma$ (transmission in “wave description”)

backscattering: $R_\sigma^\dagger L_\sigma$, $L_\sigma^\dagger\, R_\sigma$ with scattering strength depending on $V_{{\rm dis},2k_F}$ (reflection)

Microscopic origin of electrical resistance

- backscattering $(R_\sigma^\dagger L_\sigma,\ L_\sigma^\dagger R_\sigma)$ in momentum space:





- disorder-induced backscattering in 1D channels
  ⇒ origins of electrical resistance and dissipation in electronic devices
- at low T: Anderson localization of carriers in a long conductor
  - exception: edge transport in quantum Hall states (topological protection)
    ⇒ remarkable quantization of conductance as a new standard of basic unit
    von Klitzing, Annu. Rev. Condens. Matter Phys. 8, 13 (2017)

Büttiker formula

- extending the 2-terminal formula to multiterminal devices:

$$I = \frac{2e}{h}\bar{T}(\mu_1 - \mu_2)$$

$$\rightarrow \; I_i = \frac{2e}{h}\sum_j(\bar{T}_{j\leftarrow i}\,\mu_i - \bar{T}_{i\leftarrow j}\,\mu_j)$$

- I_i : net current flowing out of the terminal i
- $\bar{T}_{j\leftarrow i}$: electron transferred from terminal i to j
- relating the multiterminal conductance of a mesoscopic conductor to its scattering properties (recall the introduced scattering matrix)
- without asking underlying scattering mechanism(s)

Büttiker formula

- at low T, for multiterminal devices:

$$I_i = \frac{2e}{h} \sum_j (\bar{T}_{j\leftarrow i}\ \mu_i - \bar{T}_{i\leftarrow j}\ \mu_j)$$

- local chemical potential set by voltages:

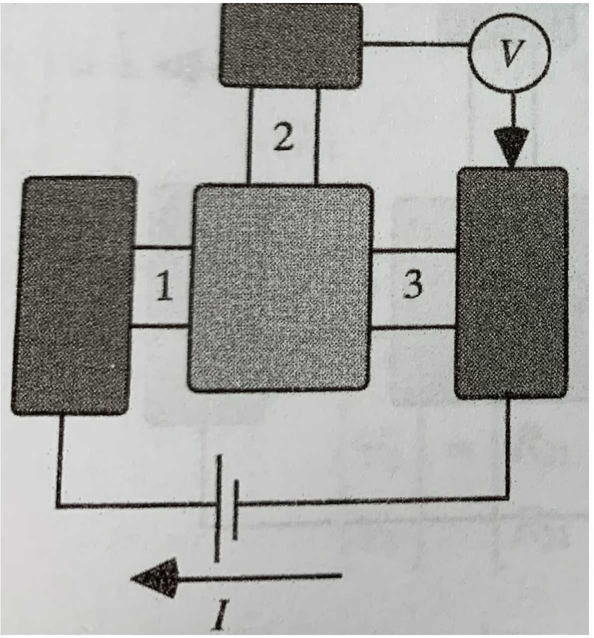
$$I_i = \sum_j (G_{ji} V_i - G_{ij} V_j) \text{ with } G_{ij} = \frac{2e^2}{h} \bar{T}_{i\leftarrow j}$$ **[note] typo before Eq. (6.71)**

  - simplified with a sum rule: $\sum_j G_{ji} = \sum_j G_{ij}$ (to ensure zero current for identical V_j)

$$\Rightarrow I_i = \sum_j G_{ij} (V_i - V_j)$$

  - description in terms of measured current and voltage without involving underlying microscopic transmission or scattering mechanism(s)

Application of the Büttiker formula



- making use of $I_i = \sum_j G_{ij}(V_i - V_j)$ at low T
  - simplified by setting one of the voltages to zero
  - simplified further with the Kirchhoff's law: $\sum_j I_j = 0$
- 3-terminal device as an example:

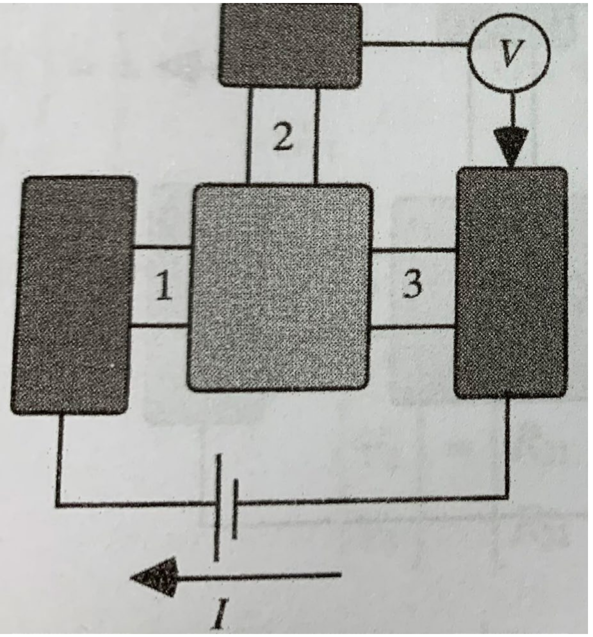
  Q: given an external current I flowing from 3 to 1, measuring V between probes 2 & 3, what is the resistance V/I ?
- from Büttiker formula:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} G_{12}+G_{13} & -G_{12} & -G_{13} \\ -G_{21} & G_{21}+G_{23} & -G_{23} \\ -G_{31} & -G_{32} & G_{31}+G_{32} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

- let $V_3 = 0$, and we know I_3 from $I_1 + I_2 + I_3 = 0$:

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} G_{12}+G_{13} & -G_{12} \\ -G_{21} & G_{21}+G_{23} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

3-terminal device



Q: what is the resistance V/I?

- inverting the matrix equation:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \boldsymbol{R} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} G_{12} + G_{13} & -G_{12} \\ -G_{21} & G_{21} + G_{23} \end{pmatrix}^{-1} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

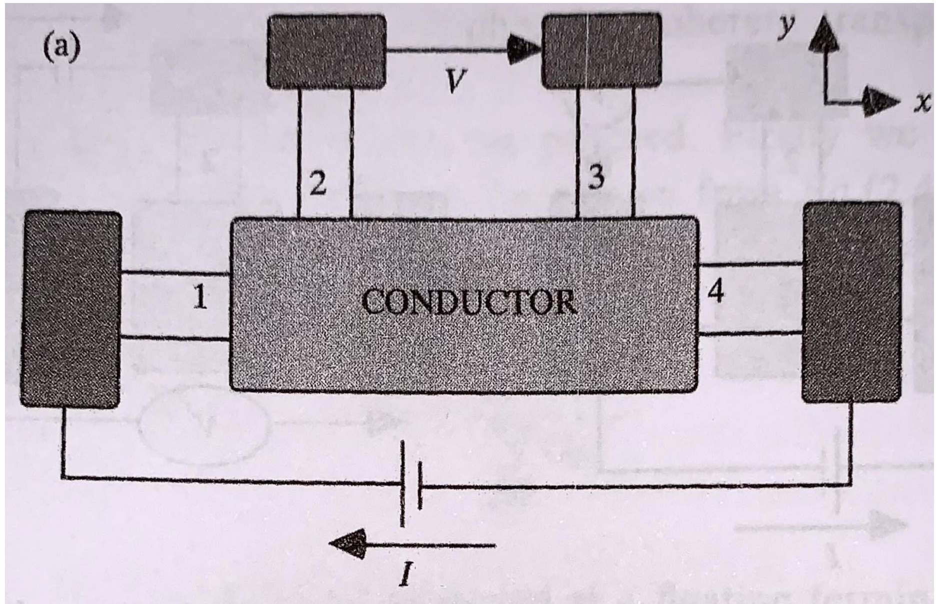
- the matrix can be inverted straightforwardly
- expressing V_1, V_2 in terms of matrix elements of ***R*** and I_1, I_2 :

$$V_1 = R_{11} I_1 + R_{12} I_2, \quad V_2 = R_{21} I_1 + R_{22} I_2$$

- V/I in terms of matrix element(s) of ***R*** (which can be expressed in terms of G_{ij}):

$$\frac{V}{I} = \left.\frac{-V_2}{-I_1}\right|_{I_2=0} = R_{21}$$

4-terminal device



- Q: given external current I from 4 to 1, measuring V between probes 2 & 3, what is the 4-terminal resistance V/I ?
- again, we have freedom to set $V_4 = 0$, and we know $I_4 = -(I_1 + I_2 + I_3)$:

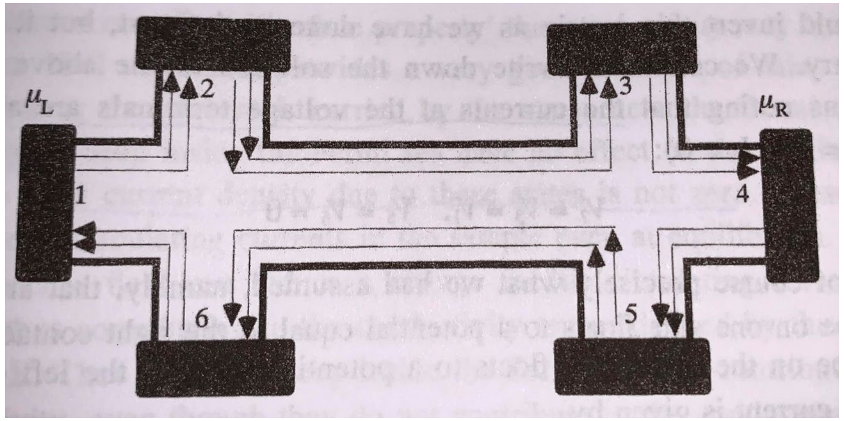
$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} G_{12}+G_{13} & -G_{12} & -G_{13} \\ -G_{21} & G_{21}+G_{23} & -G_{23} \\ -G_{31} & -G_{32} & G_{31}+G_{32} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \to \boldsymbol{R}^{-1} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

- V/I in terms of matrix element of $\boldsymbol{R}$:

$$\frac{V}{I} = \frac{V_3 - V_2}{-I_1}\bigg|_{I_2=I_3=0} = R_{21} - R_{31}$$

Edge conduction in quantum Hall states

- 6-terminal device in a quantum Hall state with M edge modes



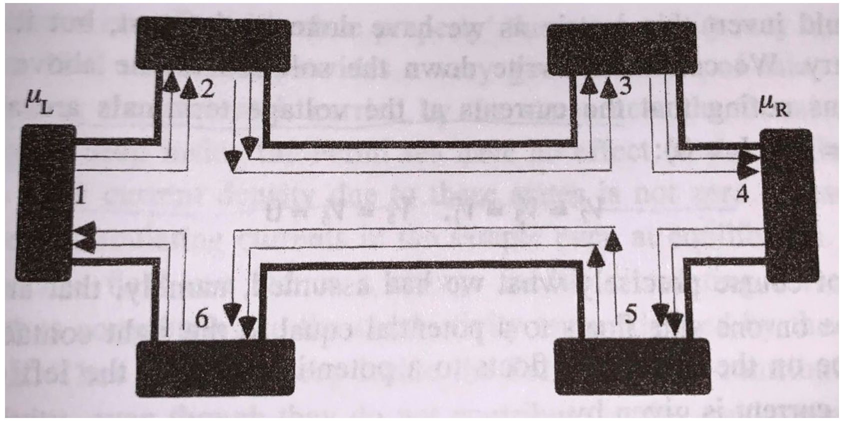
- since the bulk is gapped, only (gapless) edge modes can carry current:

$G_{ij} = \frac{2e^2}{h} M$, for $(i \leftarrow j) = (1 \leftarrow 6), (2 \leftarrow 1), (3 \leftarrow 2), (4 \leftarrow 3), (5 \leftarrow 4), (6 \leftarrow 5)$

$G_{ij} = 0$, otherwise

$\Rightarrow$ simplifying the conductance matrix in $I_i = \sum_j G_{ij}(V_i - V_j)$

Edge conduction



$I_i = \sum_j G_{ij}(V_i - V_j)$

- we set $V_4 = 0$: $\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_5 \\ I_6 \end{pmatrix} = \begin{pmatrix} G_c & 0 & 0 & 0 & -G_c \\ -G_c & G_c & 0 & 0 & 0 \\ 0 & -G_c & G_c & 0 & 0 \\ 0 & 0 & 0 & G_c & 0 \\ 0 & 0 & 0 & -G_c & G_c \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_5 \\ V_6 \end{pmatrix}$, $\quad \mathrm{G_c} = \frac{2e^2}{h} M$
- inverting the matrix could give solutions, but it is unnecessary
- we note that currents at the voltage terminals are zero: I_2=I_3=I_4=I_5=0

  $\Rightarrow V_2 = V_3 = V_1,\ V_5 = V_6 = 0,\ I_1 = G_c V_1$
- longitudinal resistance: $R_L = \frac{V_2 - V_3}{I_1} = \frac{V_6 - V_5}{I_1} = 0$, transport without dissipation!
- Hall resistance: $R_H = \frac{V_2 - V_6}{I_1} = \frac{V_3 - V_5}{I_1} = \frac{h}{2e^2 M}$, experimentally very precise!

Outline

- Review of useful concepts from quantum mechanics
- Quantum transport in mesoscopic systems
  - Landauer-Büttiker formalism (single-particle description)
    - conductance quantization in ballistic systems
    - Landauer formula for an imperfect conductor
    - Büttiker formula for multiterminal devices
    - application
  - interacting systems (beyond single-particle regime)
    - interacting electrons in 1D: Tomonaga-Luttinger liquid
    - impurities (weak and strong)
    - effects of spin-orbit-coupling

Incorporating electron-electron (*e*-*e*) interaction in 1D

- only electrons near the Fermi level participates in transport:

$$\psi \approx e^{ik_F x} R + e^{-ik_F x} L \quad \text{(spinless for simplicity)}$$

- effective theory in a 1D channel:

$$H_{\mathrm{kin}} + H_{\mathrm{int}}$$

- kinetic energy (linearized spectrum):

$$H_{\mathrm{kin}} + H_{\mathrm{int}} = -i\hbar v_F \int dx \, \left(R^{\dagger} \partial_x R - L^{\dagger} \partial_x L \right)$$

- (screened) Coulomb interaction between electrons

$$H_{\mathrm{int}} = \int dx \, V_{\mathrm{ee}}(x) \, \rho(x)\rho(x) \approx \int dx \left\{ g_2 \left(R^{\dagger} R L^{\dagger} L \right) + \frac{g_4}{2} \left[\left(R^{\dagger} R \right)^2 + \left(L^{\dagger} L \right)^2 \right] \right\}$$

- going beyond the single-particle regime => cannot be diagonalized!

Tomonaga-Luttinger liquid (TLL or LL)

- bosonization of the right- and left-moving electrons

$$R = \frac{U_R}{\sqrt{2\pi a}}\, e^{i[-\phi(x)+\theta(x)]},\ L = \frac{U_L}{\sqrt{2\pi a}}\, e^{i[\phi(x)+\theta(x)]}$$

- ϕ, θ: bosonic fields fulfilling the commutation relation:

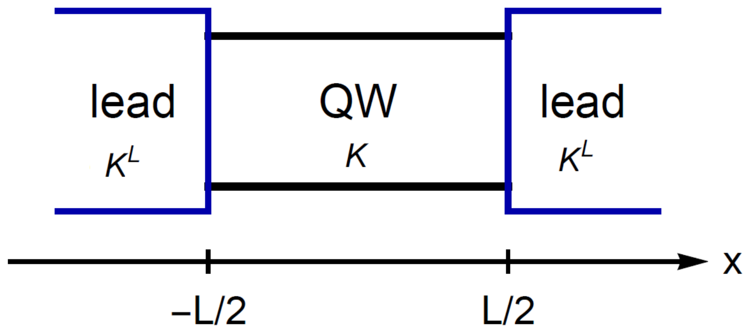
$$[\phi(x), \theta(x')] = \frac{i\pi}{2}\,\mathrm{sign}(x' - x)$$

- effective theory (mapping interacting fermions to free bosons)

$$H_{\mathrm{kin}} + H_{\mathrm{int}} = \frac{\hbar u}{2\pi}\int dx\left[\frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2\right],\quad K \equiv \left(\frac{2\pi\hbar v_F + g_4 - g_2}{2\pi\hbar v_F + g_4 + g_2}\right)^{\frac{1}{2}}$$

  - quadratic Hamiltonian ⇒ using TLL model to compute physical quantities (not here)
  - interaction strength encoded in the parameter K
    - $K = 1$: free fermions (i.e., Fermi liquid = FL)
    - $K < 1$ ($K > 1$): repulsive (attractive) interaction

Transport in clean 1D interacting systems



- clean wires connected to leads: ballistic conductance $G = \frac{2e^2}{h} \times K^L$

  Maslov and Stone, PRB 52, R5539 (1995); Ponomarenko, PRB 52, R8666 (1995); Safi and Schulz, PRB 52, R17040 (1995)

- physical meaning of contact resistance (from the last section):
  - from the process where electron wave packet from 3D reservoir gets back scattered when trying to enter the narrow conduction modes in a Q1D conductor
  - no information about *e-e* interaction within the conductor!

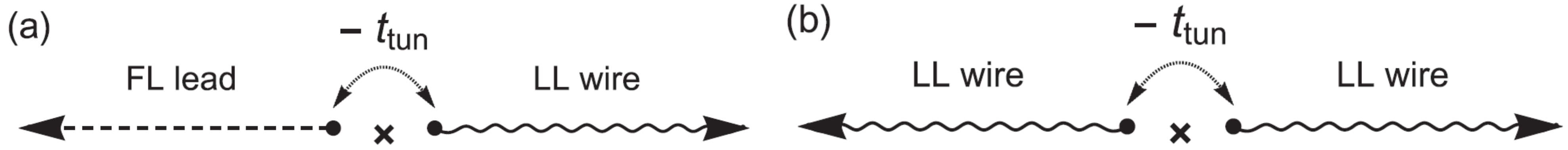
- Q: can there still be transport features coming from *e-e* interaction in the conductor?

  Yes! we need some backscattering within the conductor

Effects of impurities in 1D

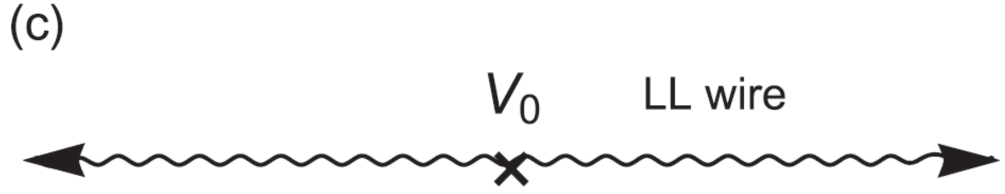
- different modeling according to their strength and positions
- strong impurities:

  acting as tunnel barriers, either at the boundary or inside the conductor

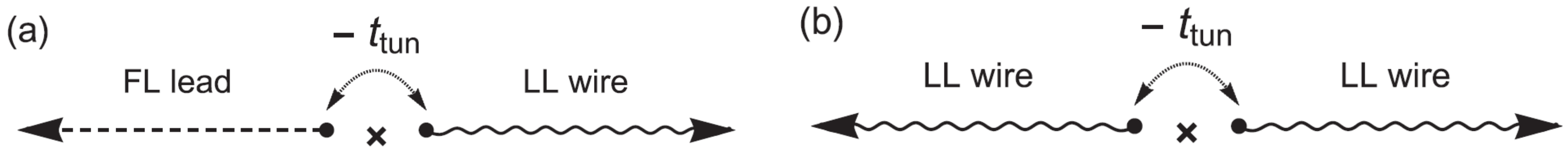


- barrier between LL wire and LL wire or between LL and FL lead
- weak impurities:

  acting as a potential perturbation



Impurities as tunnel barriers



- current through tunneling: $H_{\text{tun}} = -t_{\text{tun}} \int dx\, \delta(x)\, \psi_<^\dagger(x)\, \psi_>(x) + h.c.$

$$\frac{dI_{\text{tun}}}{dV} \propto \begin{cases} V^{\frac{1}{K}-1} & \text{(boundary barrier)} \\ V^{\frac{2}{K}-2} & \text{(interior barrier)} \end{cases}$$

  - power-law (differential) conductance with an exponent depending on impurity position and interaction strength (K=1 gives linear response for FL)
    Kane and Fisher, PRB 46, 15233 (1992)

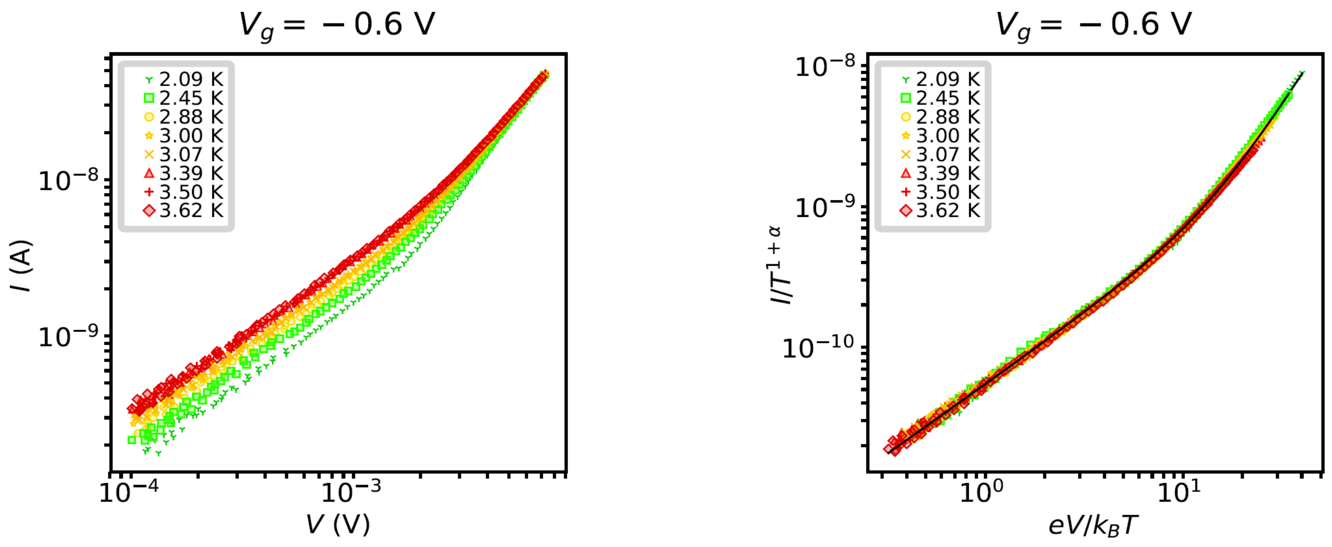
- universal scaling formula for temperature T and bias V:
  - observed in carbon nanotubes
    Bockrath et al., Nature 397, 598 (1999)

$$I = I_0 T^{1+\alpha} \sinh\left(\frac{\gamma eV}{2k_B T}\right) \left| \Gamma\left(1 + \frac{\alpha}{2} + \frac{i\gamma eV}{2\pi k_B T}\right) \right|^2$$

Universal scaling behavior in transport

$$I = I_0 T^{1+\alpha} \sinh\left(\frac{\gamma eV}{2k_B T}\right)\left|\Gamma\left(1+\frac{\alpha}{2}+\frac{i\gamma eV}{2\pi k_B T}\right)\right|^2$$

- I-V curves at different T collapse onto a single curve upon rescaling
- observation in InAs nanowires

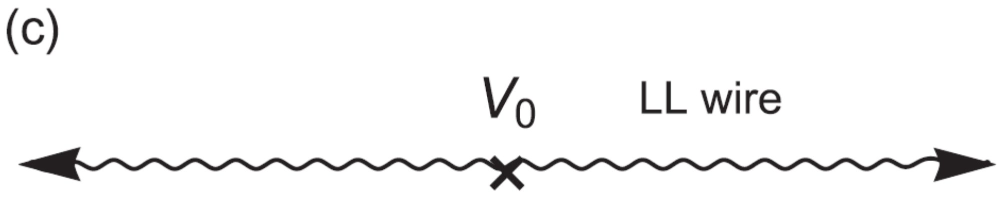


Sato et al., PRB 99, 155304 (2019)

Impurities as potential perturbation

- isolated impurity at $x = 0$:

  $H_{\text{imp}} = V_0 \int dx\, \delta(x)\, \rho(x)$



- backscattering caused by impurities: conductance correction

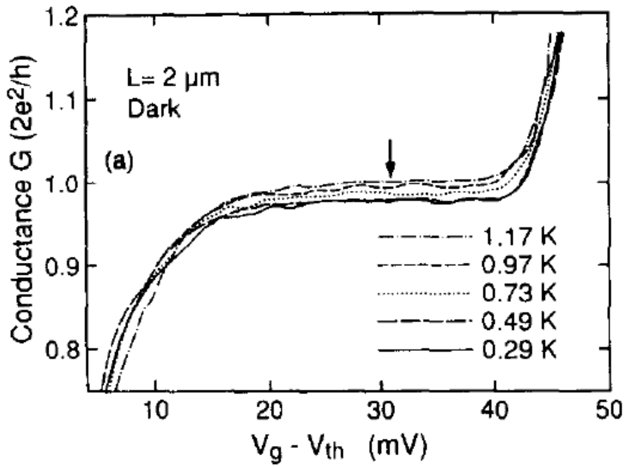
$$G = \frac{e^2}{h} + \delta G \text{ with } \delta G < 0 \text{ and } |\delta G| \propto V^{2-2K} \text{ or } |\delta G| \propto T^{2-2K}$$

  - power-law correction with a scaling exponent

Kane and Fisher, PRB 46, 15233 (1992)

- uniform reduction of conductance in GaAs wires

Tarucha et al., Sol. State. Commun. 94, 413 (1995)



General transport features in interacting systems

- backscattering effect enhanced by *e-e* interaction
  - deviation from ballistic conductance increases with interaction strength
  - $K \to 1$: usual formula for noninteracting systems (Fermi liquid)
- transport features for Tomonaga-Luttinger liquid
  - universal scaling formula
  - power-law conductance (correction)
  - interaction strength in nanodevices deduced from measurements
- Anderson localization by potential disorder:

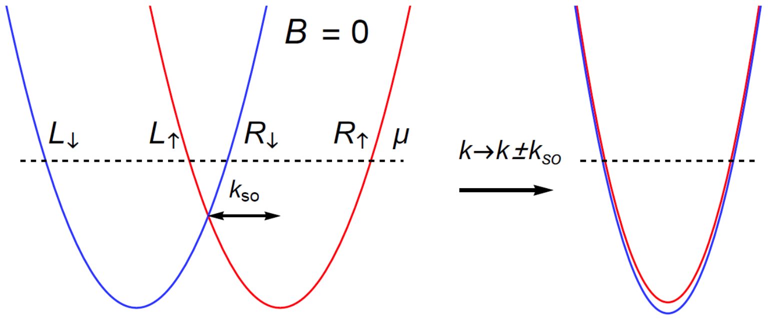
$$H_{\mathrm{dis}} = \int dx\, V_{\mathrm{dis}}(x)\, \rho(x)$$

  - *e-e* interaction enhances the tendency towards localization in 1D, with higher localization temperature and shorter localization length

    Giamarchi et al., PRB 37, 325 (1988)

Effects of spin-orbit coupling (SOC)

- Rashba SOC term in 1D semiconductors: $H_{\mathrm{R,1D}} = \alpha_R \sigma^y k_x$:
  - linear-in-momentum term can be gauged away in strict 1D



$\Rightarrow$ no spin-orbit effects on charge transport
Braunecker et al., PRB 82, 045127 (2010)

- no interaction effect in 1D clean systems
  Maslov and Stone, PRB 52, R5539 (1995); Ponomarenko, PRB 52, R8666 (1995);
  Safi and Schulz, PRB 52, R17040 (1995)
- finite width of realistic wires: higher transverse subbands in Q1D
  $\Rightarrow$ unlike strict 1D, SOC cannot be completely removed
- disorder or charge impurities in realistic wires

Spin-orbit effects on energy spectrum in Q1D wires

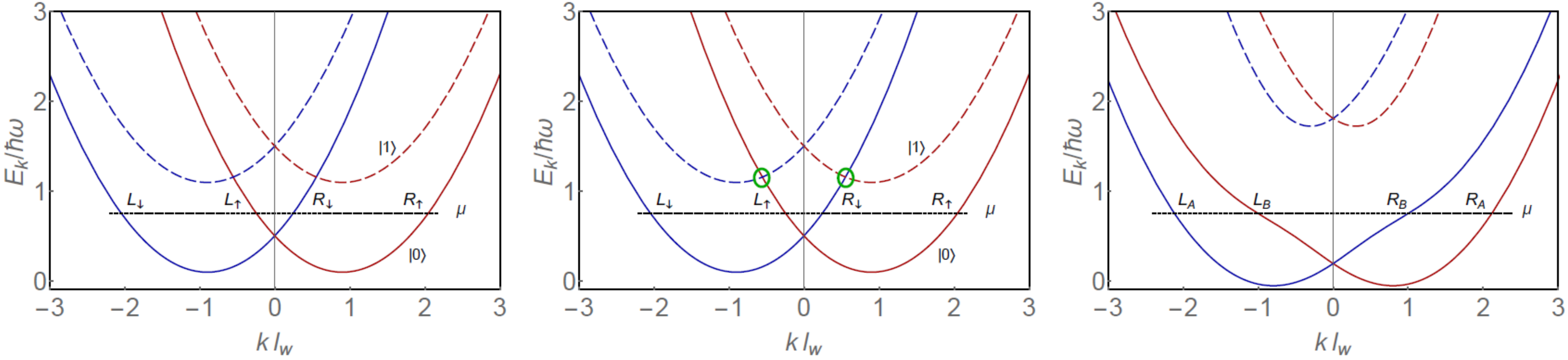
- Q1D wires $\parallel x$ with transverse subband index n:

$$H = \frac{\hbar^2 k_x^2}{2m} + \hbar\omega(n + \frac{1}{2}) + H_{\mathrm{R}}$$

- Rashba SOC term:

$$H_{\mathrm{R}} = \alpha_R(\sigma^y k_x - \sigma^x k_y)$$

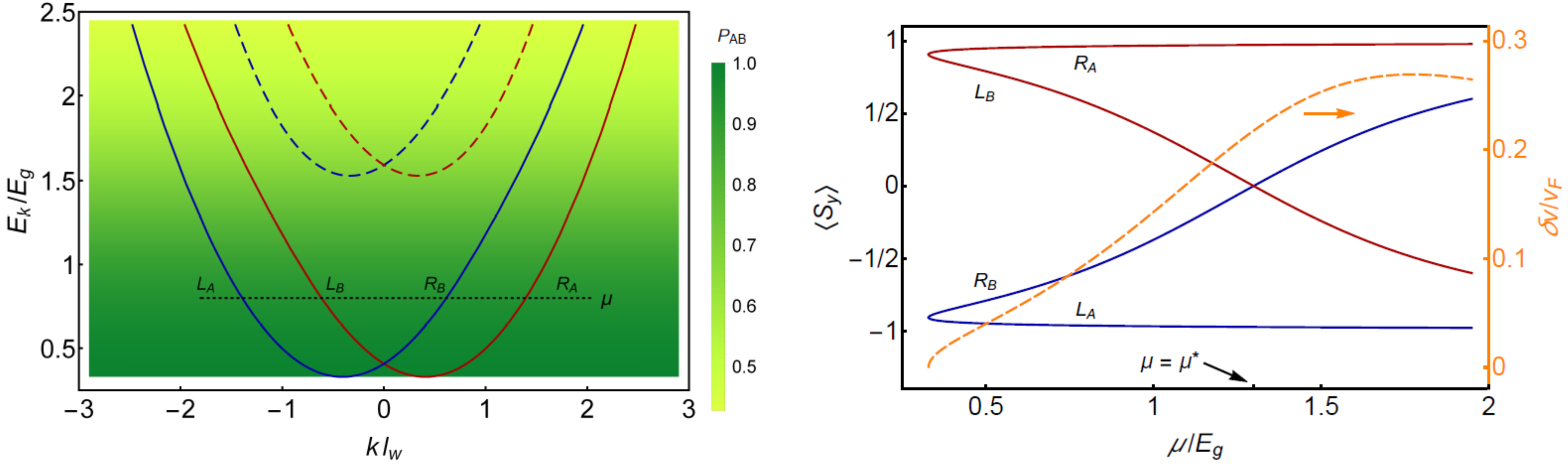
- $\sigma^y k_x$ term: shifting parabolic dispersion by $k_{\mathrm{so}} = m|\alpha_R|/\hbar^2$
- $\sigma^x k_y$ term: mixing opposite spin states of neighboring subbands $|n\rangle = |0\rangle,\ |1\rangle$



$\Rightarrow$ band distortion $\delta v = v_A - v_B$ and mixing of up- and down-spins

Spectrum and spin orientation of a SOC wire

- SOC admixes the opposite spin states of neighboring subbands
  - band distortion $\delta v = v_A - v_B$: distinct Fermi velocities of the two branches
  - spin orientation of electrons depends on chemical potential μ



- Backscattering on charge impurities between right- and left- movers:
  $P_{AB} = |\langle R_A | L_B \rangle| = |\langle R_B | L_A \rangle|$

Interacting 1D channel with spin

- low T: only electrons near Fermi level matter: $\psi_\sigma \approx e^{ik_F x} R_\sigma + e^{-ik_F x} L_\sigma$
- kinetic energy (linearized):

$$H_{\text{kin}} = -i\, \hbar v_F \sum_\sigma \int dx\, (\, R_\sigma^\dagger\, \partial_x\, R_\sigma - L_\sigma^\dagger\, \partial_x\, L_\sigma)$$

- *e*-*e* interaction:

$$H_{\text{int}} = \int dx\, V_{\text{ee}}(x)\, \rho(x)\rho(x) \approx \sum_\sigma V_{\text{ee},0} \int dx\, \left\{ R_\sigma^\dagger R_\sigma L_\sigma^\dagger L_\sigma + \frac{1}{2}\left[\left(R_\sigma^\dagger R_\sigma\right)^2 + \left(L_\sigma^\dagger L_\sigma\right)^2 \right] \right\}$$

- bosonization:

$$R_\sigma = \frac{U_{R\sigma}}{\sqrt{2\pi a}} e^{i[-\phi_c(x)+\theta_c(x)-\sigma\phi_s(x)+\sigma\theta_s(x)]/\sqrt{2}},\ L_\sigma = \frac{U_{R\sigma}}{\sqrt{2\pi a}} e^{i[\phi_c(x)+\theta_c(x)+\sigma\phi_s(x)+\sigma\theta_s(x)]/\sqrt{2}}$$

- spinful (Tomonaga-)Luttinger liquid with two charge (c) and spin (s) sectors

$$H_{\text{kin}} + H_{\text{int}} = \sum_{\nu=c,s} \int \frac{\hbar dx}{2\pi} \left\{ u_\nu K_\nu [\partial_x \theta_\nu]^2 + \frac{u_\nu}{K_\nu} [\partial_x \phi_\nu]^2 \right\}$$

- charge-spin separation in usual 1D wires (negligible spin-orbit coupling)

Spin-orbit effects on Q1D wires

- Q1D + SOC: band distortion

  ⇒ causing a charge-spin mixing term in the Hamiltonian

$$H_{\mathrm{so}} = \delta v \int \frac{\hbar dx}{2\pi} \{[\partial_x \phi_c(x)][\partial_x \theta_s(x)] + [\partial_x \phi_s(x)][\partial_x \theta_c(x)]\}$$

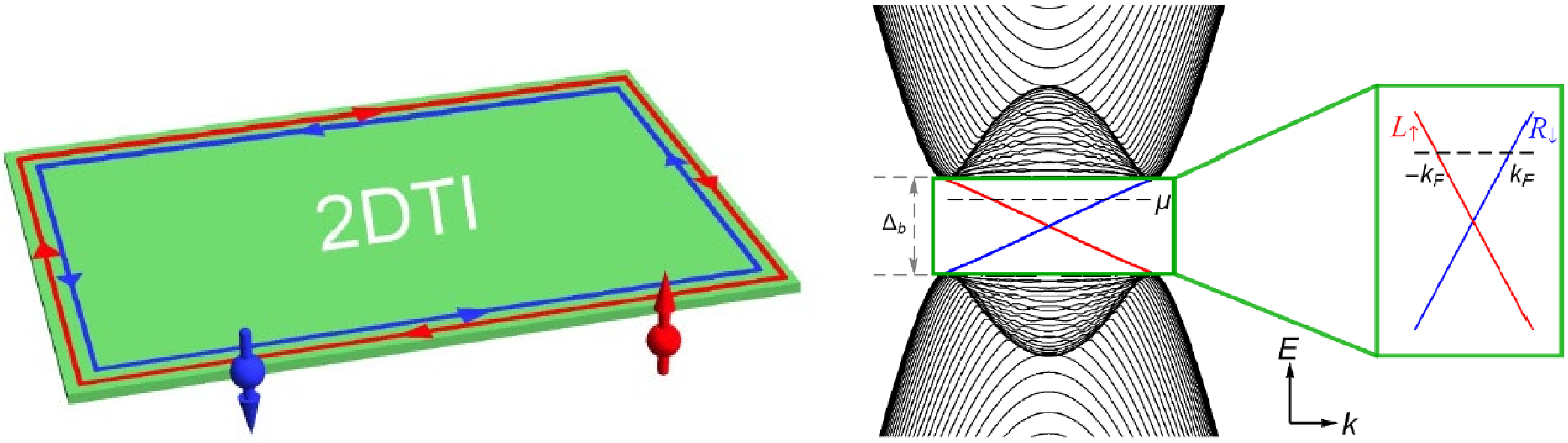
- Q1D + SOC + impurities: new transport features

  ⇒ power-law conductance and universal scaling formula with scaling exponents depending on *e*-*e* interaction and spin-orbit-induced band distortion

  Sato et al., PRB 99, 155304 (2019); Hsu et al., PRB 100, 195423 (2019)

if time permits …

Quantum spin Hall insulator (QSHI) and edge states



Hsu et al., SST 36, 123003 (2021)

- also called two-dimensional topological insulator (2DTI)
- gapless edge states protected by the bulk topology
- helical nature: spin-momentum locking
- interacting electrons in one dimension

Interacting electrons in 1D edge channels

- effective theory for electrons in a helical edge (coordinate r):

$$H_{\text{hel}} = H_{\text{kin}} + H_{\text{ee}}$$

- kinetic energy

$$H_{\text{kin}} = -i\hbar v_F \int dr \left(R^\dagger \partial_r R - L^\dagger \partial_r L \right)$$

- (screened) Coulomb interaction between electrons

$$H_{\text{ee}} = g_2 \int dr R^\dagger R L^\dagger L + \frac{g_4}{2} \int dr \left[\left(R^\dagger R\right)^2 + \left(L^\dagger L\right)^2 \right]$$

Helical Tomonaga-Luttinger liquid (hTLL)

- effective edge theory

$$H_{\mathrm{hel}} = \frac{\hbar u}{2\pi} \int dr \left[\frac{1}{K} (\partial_r \phi)^2 + K (\partial_r \theta)^2 \right] \quad K \equiv \left(\frac{2\pi\hbar v_F + g_4 - g_2}{2\pi\hbar v_F + g_4 + g_2} \right)^{\frac{1}{2}}$$

- bosonization of the right- and left-moving edge modes

$$R_{\downarrow} = \frac{U_R}{\sqrt{2\pi a}} e^{ik_F r} e^{i[-\phi(r)+\theta(r)]},$$
$$L_{\uparrow} = \frac{U_L}{\sqrt{2\pi a}} e^{-ik_F r} e^{i[\phi(r)+\theta(r)]}.$$

- interaction strength encoded in the parameter K (K=1: no interaction)
- how to extract its value from a real sample?

Spectroscopic signatures: local density of states (DOS)

- local DOS as functions of T and $\epsilon = E - E_F$

$$\rho_{\mathrm{TLL}}(\epsilon, T) \propto T^{\alpha} \cosh\left(\frac{\epsilon}{2k_B T}\right) \left| \Gamma\left(\frac{1+\alpha}{2} + i\frac{\epsilon}{2\pi k_B T}\right) \right|^2$$

Bockrath et al., Nature 397, 598 (1999)

- $\rho_{\mathrm{TLL}}/T^{\alpha}$ depends only on the ratio of ϵ/T for any ϵ and T
  $\Rightarrow$ universal scaling behavior
- the exponent α depends on the interaction strength
- for a helical edge, $\alpha = (K + 1/K - 2)/2$
  $\Rightarrow$ stronger interaction gives smaller K and larger α

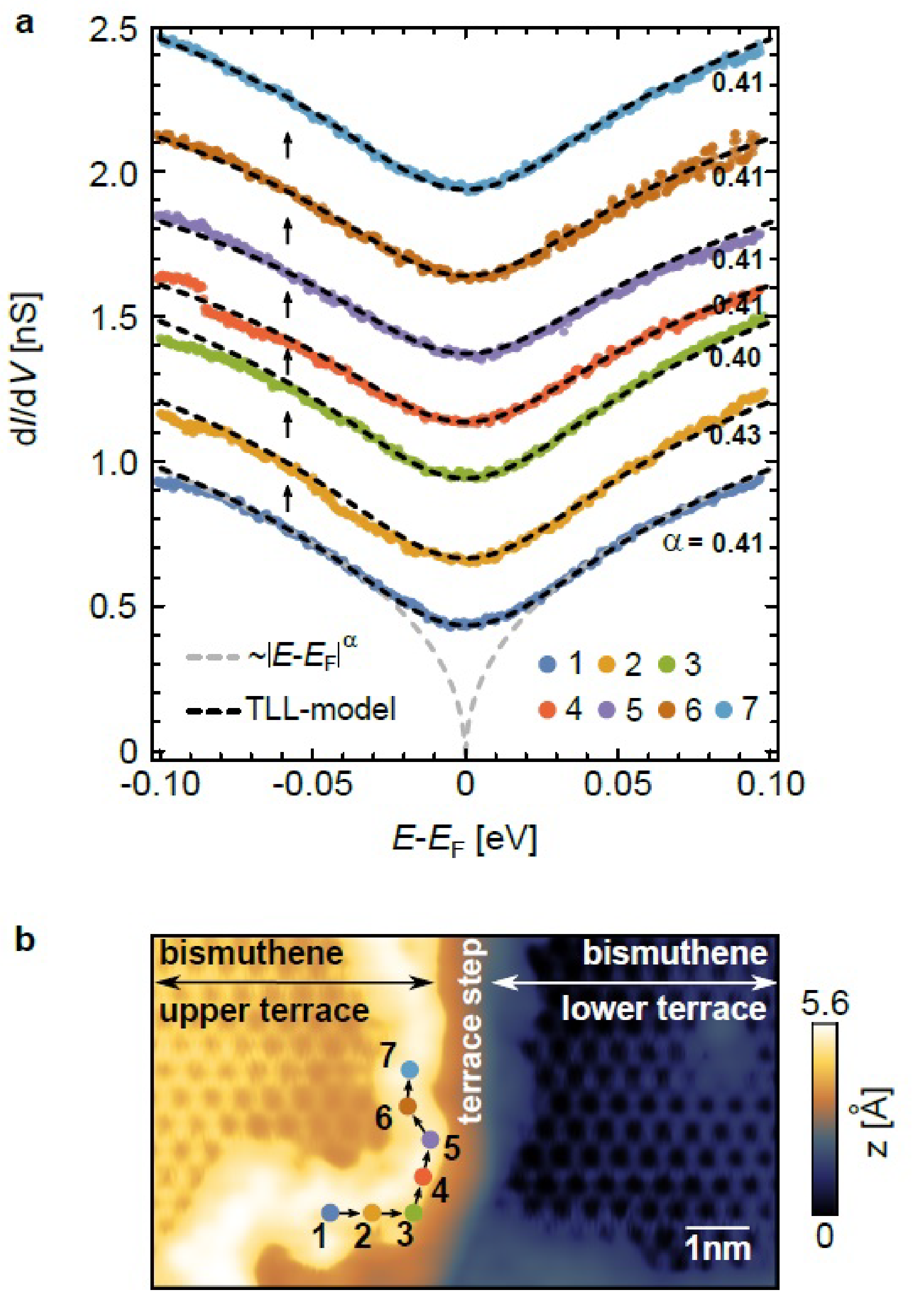
Experimental observations

- local DOS of the edge mode

$$T^\alpha \cosh\left(\frac{\epsilon}{2k_BT}\right)\left|\Gamma\left(\frac{1+\alpha}{2}+i\frac{\epsilon}{2\pi k_BT}\right)\right|^2.$$

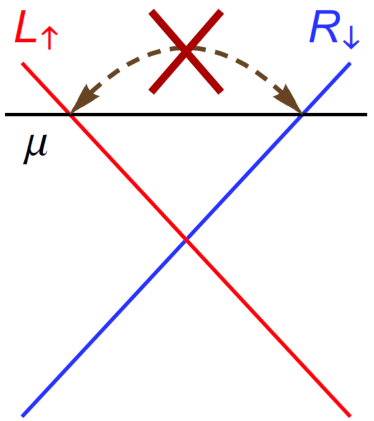
Stühler et al., Nat. Phys. 16, 47 (2019)

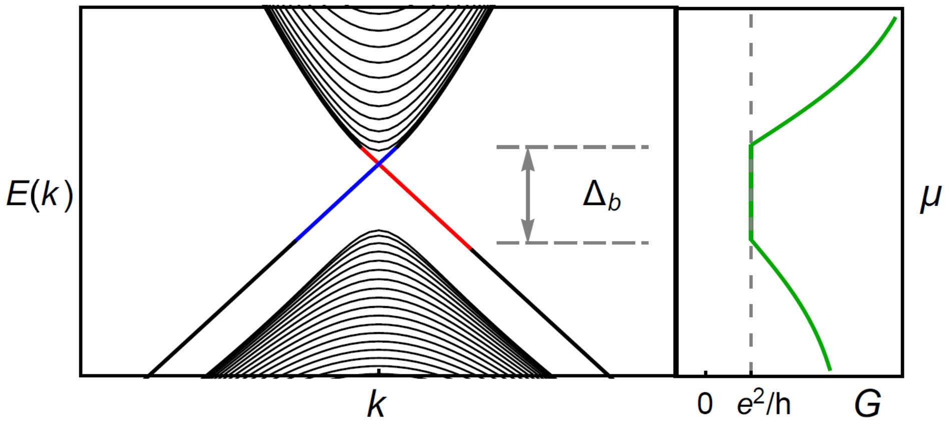
- asymptotic behavior
  - $\rho \propto |\epsilon|^\alpha$ for $|\epsilon| \equiv |E-E_F| \gg k_BT$
  - $\rho \propto T^\alpha$ for $|\epsilon| \ll k_BT$
- local DOS at the edge
  - universal scaling behavior
  - the fitted parameter value is consistent with the estimated value $K \approx$ 0.4-0.6
- spectroscopic feature for helical liquids in QSHI
- how about transport?



Edge transport in QSHI

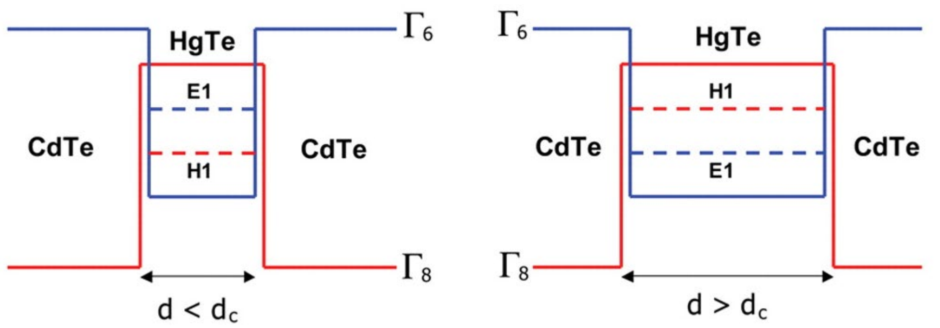
- $R_\downarrow$ and $L_\uparrow$ in helical channels:
  spin flip necessary for elastic backscattering $R_\downarrow \leftrightarrow L_\uparrow$
- Charge impurities:
  creating potential disorder $V_{\rm dis}$ but *no spin flip*: $\langle L_\uparrow | V_{\rm dis} | R_\downarrow \rangle = 0$
  $\Rightarrow$ (naive) expectation: no edge resistance





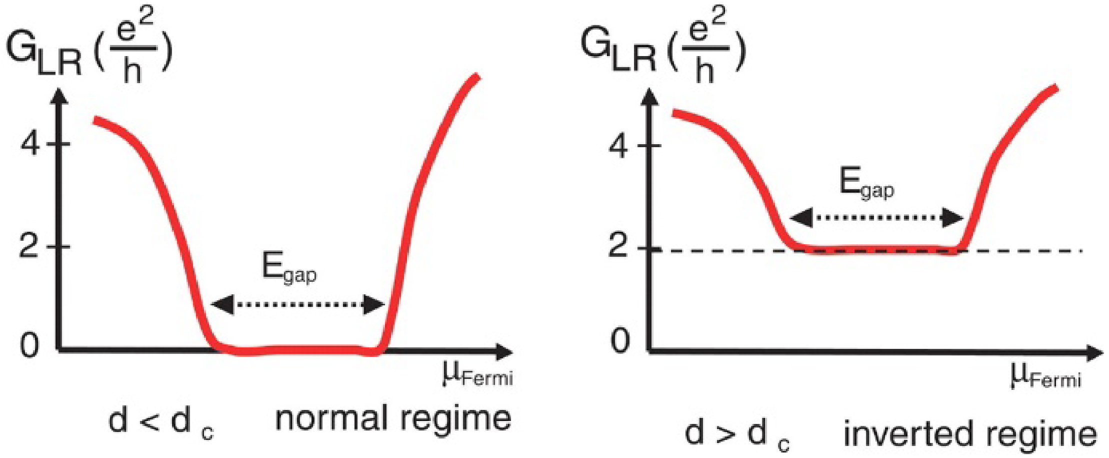
- Transport signature when the chemical potential μ is in the bulk gap $\Delta_{\rm b}$
  $\Rightarrow$ quantized edge conductance at e^2/h

HgTe/CdTe

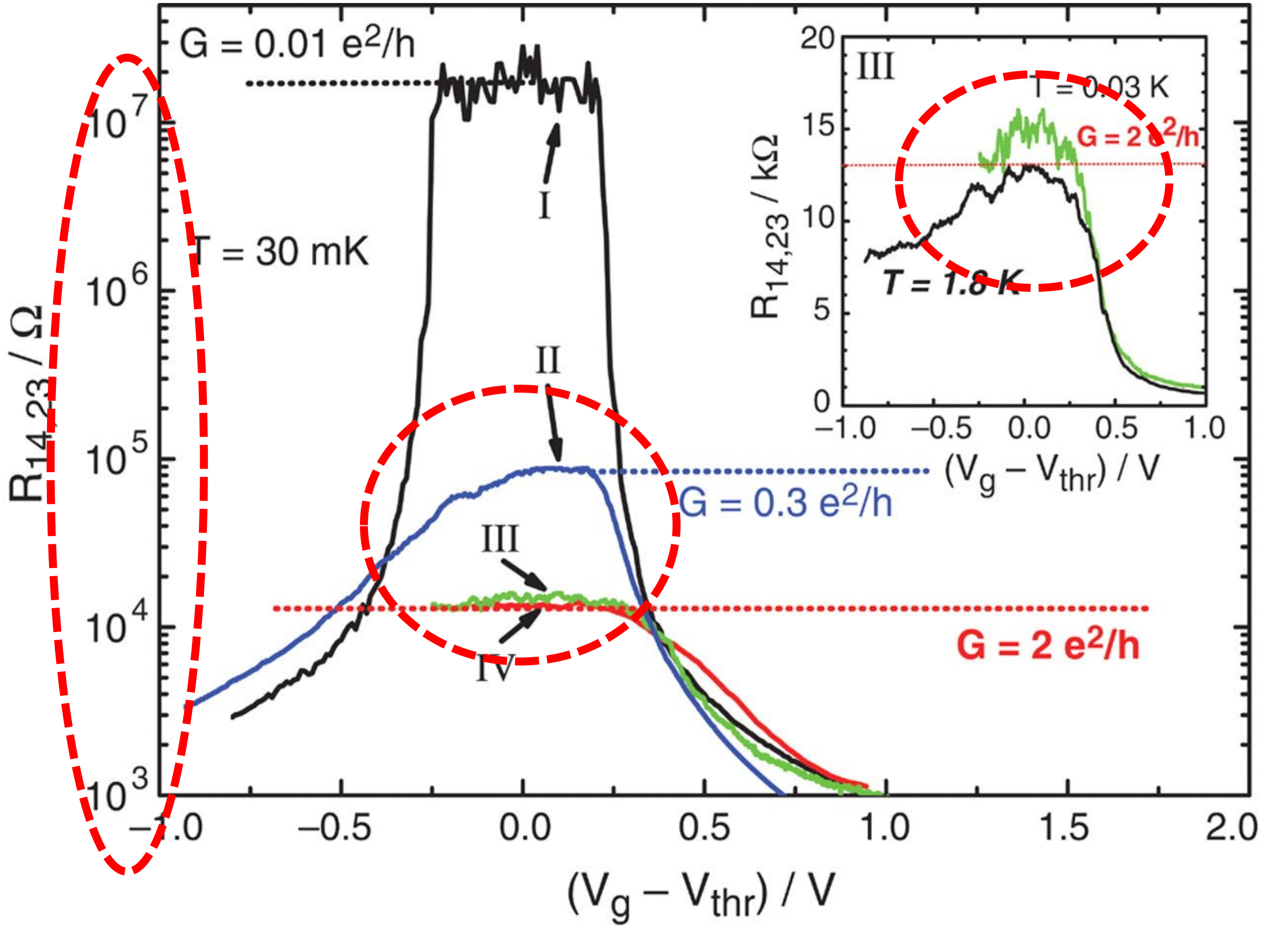


Theory: B. A. Bernevig et al. Science 314, 1757 (2006)

Bulk energy bands of CdTe/HgTe/CdTe quantum well



2-terminal measurement gives $G_{LR} = 2e^2/h$ in the topological regime, and $G_{LR} = 0$ in the trivial regime



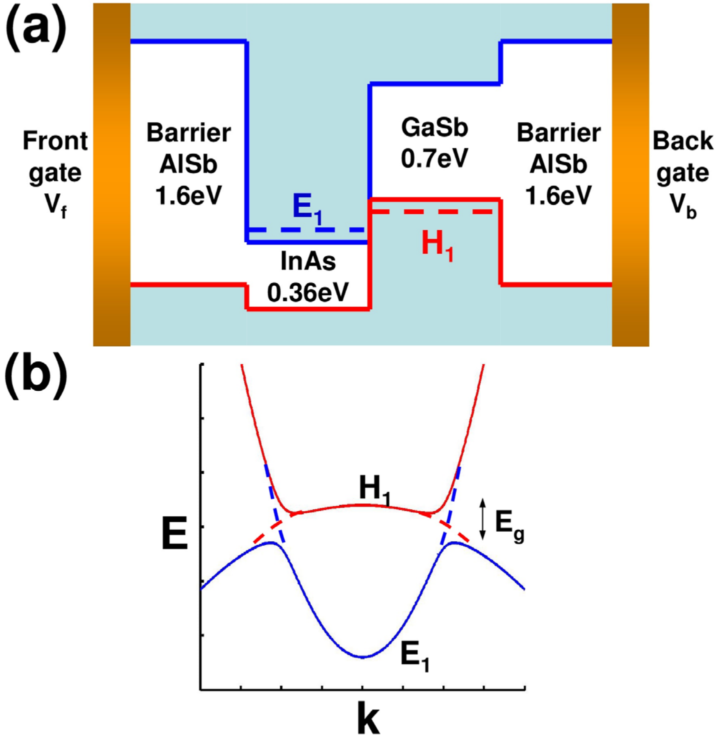
Experiment: M. König et al. Science 318, 766 (2007).

Longitudinal resistance of normal (I) and inverted (II, III, and IV) regimes as a function of the gate voltage at T = 30 mK

device sizes:

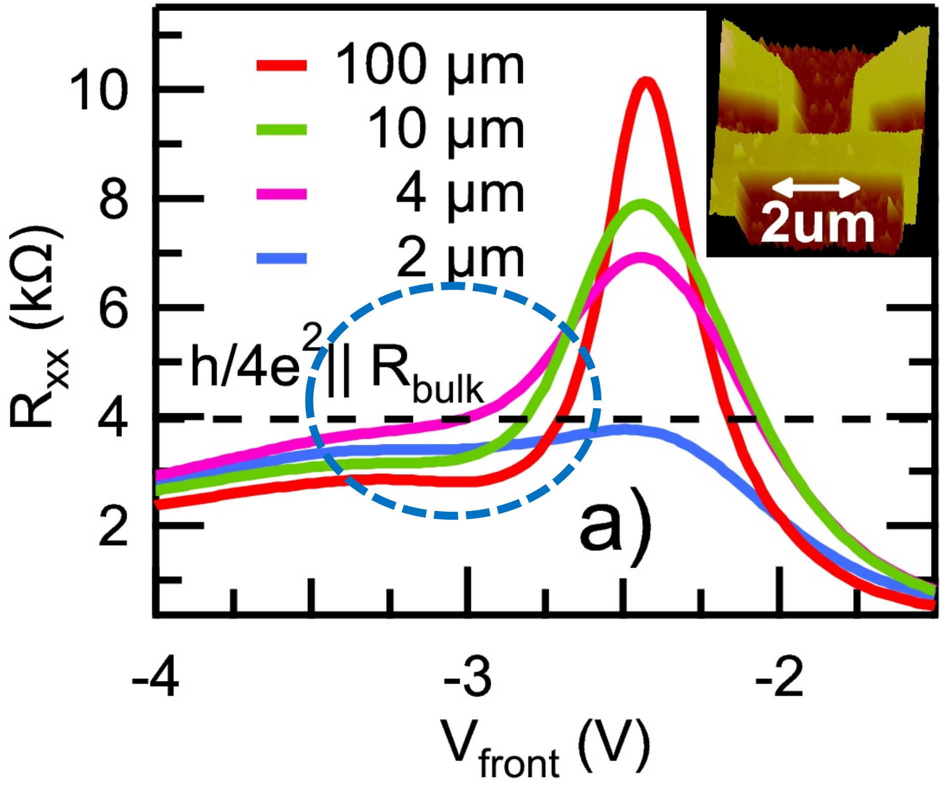
20 x 13.3 μm^2 (I and II), 1.0 x 1.0 μm^2 (III), and 1.0 x 0.5 μm^2 (IV)

InAs/GaSb



Theory: C. Liu et al., PRL 100, 236601 (2008)

Energy bands of InAs/GaSb QW



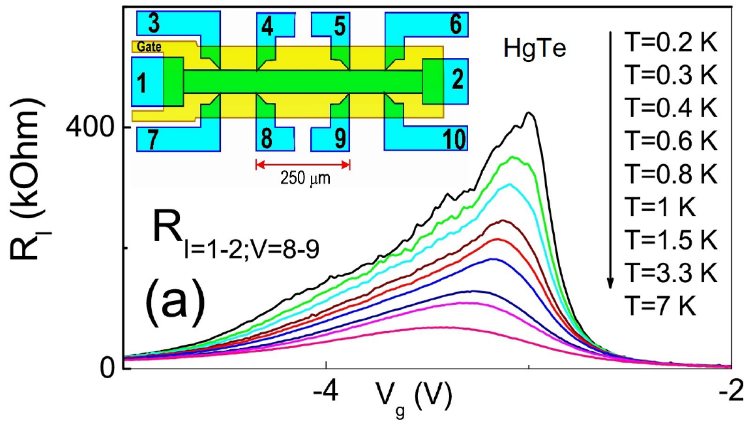
Experiment: I. Knez et al., PRL 107, 136603 (2011)

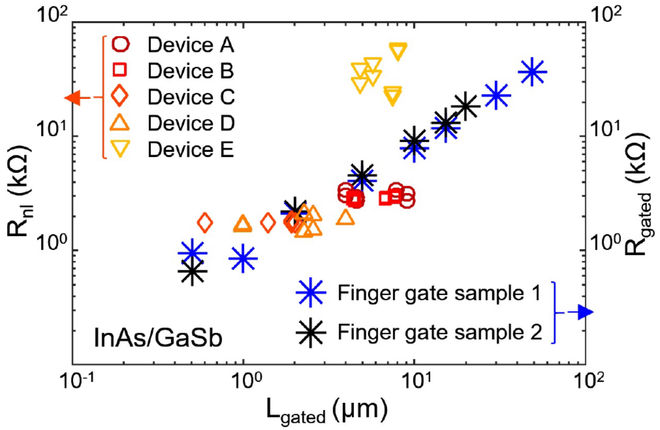
Longitudinal resistance of InAs/GaSb QW as a function of the gate voltage for various edge lengths at T = 300 mK

- theory: quantized edge conductance
- experiments did not agree!

Finite edge resistance in realistic samples

- experiments:
  - no robustly quantized conductance in larger samples
  - presence of backscattering and resistance sources





HgTe **Olshanetsky et al., PRL 114, 126802 (2015)** and InAs/GaSb **Mueller et al., PRB 96, 075406 (2017)**

- various backscattering mechanisms were proposed:
  - *e-e* interaction, SOC, noise, phonons, charge puddles, magnetic impurities, ...
- time-reversal-invariant (inelastic) or time-reversal-symmetry breaking mechanisms

Time-reversal-symmetry breaking mechanisms

TRS breaking mechanism	R or $-\delta G$	Remark
Single magnetic impurity	$\begin{cases} T^{2K-2} & \text{for } T \ll T_K \\ \text{const.} + \ln\left(\frac{\Delta_b}{k_B T}\right) & \text{for } T > T_K \end{cases}$	
Single charge impurity (with a finite magnetic field)	T^{2K-2}	
Kondo lattice (1D Kondo array)	$\begin{cases} T^{-2} & \text{for } E_{\text{pin}} < k_B T \ll \Delta_{\text{ka}}, \\ T^{2K-2} & \text{for } k_B T > \Delta_{\text{ka}} \end{cases}$	Localization at low T
Magnetic-impurity ensemble (with spin diffusion into the bulk)	$\begin{cases} e^{\Delta_{\text{rs}}/(k_B T)} & \text{for } T < T_{\text{rs}} \\ T^{2K-2} & \text{for } T > T_{\text{rs}} \end{cases}$	Localization for $K < 3/2$
Spiral-order-induced field (below spiral ordering T_s)	$\begin{cases} m_s^2 e^{\Delta_{\text{sa}}/(k_B T)} & \text{for } T < T_{\text{sa}} \\ m_s^2 T^{2K-2} & \text{for } T > T_{\text{sa}} \end{cases}$	Localization for $K < 3/2$
Magnon (below spiral ordering T_s)	$\begin{cases} \omega_m^{2K-3} & \text{for magnon emission} \\ T^{3-2K} & \text{for magnon absorption} \end{cases}$	
DNP[f] (for $K \approx 1$ and finite spin-flip rate)	$(T + \text{const.})^{-1}$	
DNP with random SOI[g] (for $K \approx 1$ and long channels)	$T^{-2/3}$	

Hsu et al., SST 36, 123003 (2021)

Time-reversal-invariant mechanisms

TRS preserving mechanism	R or $-\delta G$
1PB by $H_{\mathrm{ee},5}$ (for clean systems)	$\begin{cases} e^{-\hbar v_F k_F/(k_B T)} & \text{for } k_B T \ll \hbar v_F k_F \\ T^{2K+3} & \text{for } k_B T \gg \hbar v_F k_F \end{cases}$
1PB by $H_{\mathrm{ee},5}$ & $H_{\mathrm{imp,f}}$	T^{2K+2}
1PB by $H_{\mathrm{ee},5}$ & $H^{\mathrm{loc}}_{\mathrm{imp}}$	T^{2K+2}
1PB by $H_{\mathrm{ee},1}$ & $H_{\mathrm{imp,b}}$	T^6 for $K \approx 1$
1PB by $H_{\mathrm{ee},3}$ & $H_{\mathrm{imp,b}}$	T^4 for $K \approx 1$
2PB by $H_{\mathrm{ee},3}$ & $H_{\mathrm{imp,f}}$	T^{8K-2} Localization for $K < 3/8$.
2PB by $H_{\mathrm{ee},3}$ & $H^{\mathrm{loc}}_{\mathrm{imp}}$	T^{8K-2}
Random SOI	0
Higher-order random SOI (single scatterer)	For $K > 1/2$: $\begin{cases} T^{4K} \text{ for } T < T^*_{\mathrm{rso}} \\ T^{4K} \ln^2(k_B T/\Delta_b) \text{ for } T > T^*_{\mathrm{rso}} \end{cases}$ For $1/4 < K < 1/2$: $\begin{cases} T^{8K-2} \text{ for } T < T^*_{\mathrm{rso}} \\ T^{4K} \ln^2(k_B T/\Delta_b) \text{ for } T > T^*_{\mathrm{rso}} \end{cases}$
1PB in charge puddles (for $K \approx 1$)	Short channel (a single puddle): even valley: $\begin{cases} T^4 & \text{for } k_B T \ll \delta_d \\ T^2 & \text{for } \delta_d \ll k_B T \ll E_{\mathrm{ch}} \\ \text{const.} & \text{for } k_B T \gg E_{\mathrm{ch}} \end{cases}$ even-odd transition: $\begin{cases} T^4 & \text{for } k_B T \ll \Gamma_t \\ \text{const.} & \text{for } \Gamma_t \ll k_B T \ll \delta_d \\ T^2 & \text{for } \delta_d \ll k_B T \ll E_{\mathrm{ch}} \\ \text{const.} & \text{for } k_B T \gg E_{\mathrm{ch}} \end{cases}$ odd valley: $\begin{cases} T^4 & \text{for } T \ll T_K \\ \ln^2(T/T_K) & \text{for } k_B T_K \ll k_B T \ll \delta_d \\ T^2 & \text{for } \delta_d \ll k_B T \ll E_{\mathrm{ch}} \\ \text{const.} & \text{for } k_B T \gg E_{\mathrm{ch}} \end{cases}$ Long channel: $E_{\mathrm{ch}} \ll \delta_d$: T for $k_B T \ll \delta_d$; $E_{\mathrm{ch}} \approx \delta_d$: $1/\ln^2[\delta_d/(k_B T)]$ for $k_B T \ll \delta_d$; $E_{\mathrm{ch}} \gg \delta_d$: $\begin{cases} 1/\ln^2[\delta_d/(k_B T)] & \text{for } k_B T \ll \delta'_d \\ 1/\ln[\delta_d/(k_B T)] & \text{for } \delta'_d \ll k_B T \ll \delta_d \end{cases}$
Noise[j] (for $K \approx 1$, long channels)	Telegraph noise: $T^2 \tanh\left(\frac{E_{\mathrm{ch}}}{2k_B T}\right)$ $1/f$ noise: $\begin{cases} T^2 & \text{for } k_B T \ll E_{\mathrm{ch}} \\ T & \text{for } k_B T \gg E_{\mathrm{ch}} \end{cases}$
Acoustic longitudinal phonon	0
Transverse phonon (for $K \approx 1$)	Short channel: $\begin{cases} e^{-\hbar v_F k_F/(k_B T)} & \text{for } k_B T \ll \hbar c_{\mathrm{ph},t} k_F \\ \text{const. } T^3 + \text{const. } T^5 & \text{for } k_B T \gg \hbar c_{\mathrm{ph},t} k_F \end{cases}$ Long channel: $\begin{cases} T^3 & \text{for } k_B T \ll \hbar v_F k_F,\ k_B \theta_D \\ T^5 & \text{for } \hbar v_F k_F \ll k_B T \ll k_B \theta_D \\ T & \text{for } k_B T \gg k_B \theta_D \gg \hbar v_F k_F \end{cases}$

Reference	Notation or name in the original work
Kainaris *et al* (2014)	$g_1 \times b$ process
Wu *et al* (2006) Xu and Moore (2006) Kainaris *et al* (2014)	H_{dis} or two-particle backscattering due to quenched disorder Scattering by spatially random quenched impurities $g_3 \times f$ process (in their class of two-particle processes)
Schmidt *et al* (2012) Kainaris *et al* (2014)	$H^{\mathrm{eff}}_{V,\mathrm{int}}$ $g_3 \times b$ process (in their class of one-particle processes)
Wu *et al* (2006) Maciejko *et al* (2009) Lezmy *et al* (2012)	H'_{bs} or impurity-induced two-particle correlated backscattering H_2 or local impurity-induced two-particle backscattering g_{2p} process or two-particle scattering
Schmidt *et al* (2012) Kainaris *et al* (2014) Chou *et al* (2015)	H_{int} or inelastic backscattering of a single electron with energy transfer to another particle-hole pair g_5 process $\hat{H}_W$ or one-particle spin-flip umklapp term
Kainaris *et al* (2014) Chou *et al* (2015)	$g_5 \times f$ process (in their class of one-particle processes) $\hat{H}_W$ (same notation for clean and disordered systems)
Kainaris *et al* (2014)	$g_5 \times b$ (in their class of one-particle processes)
Lezmy *et al* (2012)	g_{ie} process or inelastic scattering
Ström *et al* (2010) Geissler *et al* (2014) Kainaris *et al* (2014) Xie *et al* (2016)	H_R or randomly fluctuating Rashba spin–orbit coupling Random Rashba spin–orbit coupling $g_{\mathrm{imp,b}}$ process Random Rashba backscattering
Kharitonov *et al* (2017)	$\hat{H}_R$ or $U(1)$-asymmetric single-particle backscattering field
Crépin *et al* (2012)	Inelastic two-particle backscattering from a Rashba impurity

Hsu et al., SST 36, 123003 (2021)